Estimation and Inference. Part II

Estimation of the Autocovariances

- An estimator for the autocovariance and autocorrelations of a process $\{y_t\}$ can be formulated as

$$\widehat{\gamma}_k = \frac{1}{T}\sum_{t=k+1}^{T}(y_t - \overline{y}_T)(y_{t-k} - \overline{y}_T) \text{ for all } k$$

$$\widehat{\rho}_k = \frac{\widehat{\gamma}_k}{\widehat{\gamma}_0} \text{ for all } k.$$

Theorem (Central Limit Theorem)

If $y_t = \psi(L)\varepsilon_t$ *where* $\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + ...$,

(i) $\sum_{j=1}^{\infty} |\psi_j| < \infty$

(ii) $\{\varepsilon_t\}$ *are iid*$(0, \sigma^2)$ *and* $\mathbb{E}(\varepsilon_t^2) < \infty$ *for all t*

then we have

$$\sqrt{T}\ (\widehat{\gamma}_k - \gamma_k) \xrightarrow{d} N(0, W),$$

as $T \to \infty$.

The variance W has a very complex on expression, and therefore have to omit it here.

Remark (General Result)

If an estimator $\widehat{\theta}_T$ of some parameter θ satisfies a central limit theorem

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \xrightarrow{d} N(0, V),$$

then the confidence interval can be written as

$$\widehat{\theta}_T \pm Z_{1-\alpha/2}\sqrt{\frac{V}{T}}.$$

(for 95% conf. interval we have $Z_{1-\alpha/2} = 1.96$)

Remark (White Noise)

- *Consider white noise process* $\{y_t\}$*. We know from the definition of the WN that* $\rho(h) = 0$ *for all* $h \geq 1$*. From the CLT we can also say*

$$\sqrt{T}\,\widehat{\rho}(h) \to N(0,1) \;\; or \;\; \widehat{\rho}(h) \sim N\left(0, \frac{1}{T}\right).$$

- *This in turn gives us confidence intervals for* $\widehat{\rho}(h)$ *,i.e.,*

$$\widehat{\rho}(h) \pm Z_{1-\alpha/2}\sqrt{\frac{1}{T}}.$$

(for 95% conf. interval we have $Z_{1-\alpha/2} = 1.96$*)*

Estimation of the AR model

- To estimate the parameters of AR(p) models we can use least squares (OLS) procedures

- For general AR(p) model $y_t = \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t$ we have that under assumption

$$\mathbb{E}[y_{t-j}\varepsilon_t] = 0,$$

for all $j = 1, ..., p$ the OLS estimator of AR (p) provide consistent estimator.

Estimation of the AR model

Example (Estimation of AR(1) model)

- Consider the following AR(1) model with mean zero

$$y_t = \phi y_{t-1} + \varepsilon_t,$$

where $|\phi| < 1$ and $\varepsilon_t \sim \text{iid}(0, \sigma^2)$. So the process is stationary and ergodic.

- The OLS estimator of ϕ is given by:

$$\widehat{\phi}_T = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2}.$$

Estimation of the AR model

Example (cont.)

- We can also rewrite the OLS estimator as

$$\widehat{\phi}_T = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} = \phi + \frac{\sum_{t=2}^{T} y_{t-1}\varepsilon_t}{\sum_{t=2}^{T} y_{t-1}^2}$$

$$\text{or} \quad \sqrt{T}\left(\widehat{\phi}_T - \phi\right) = \frac{\frac{1}{\sqrt{T}}\sum_{t=2}^{T} y_{t-1}\varepsilon_t}{\frac{1}{T}\sum_{t=2}^{T} y_{t-1}^2}$$

- We have asymptotically the following

$$\frac{1}{T}\sum_{t=1}^{T} y_t^2 \xrightarrow{p} \mathbb{E}[y_t^2] = \frac{\sigma^2}{1-\phi^2} \text{ for this we use LLN}$$

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{T} y_{t-1}\varepsilon_t \xrightarrow{d} N(0, V) \text{ for this we use CLT}$$

Estimation of the AR model

Example (cont.)

- The expression for the variance V

$$V[y_{t-1}\varepsilon_t] = \mathbb{E}[y_{t-1}^2\varepsilon_t^2] = \sigma^2\mathbb{E}[y_{t-1}^2] = \frac{\sigma^4}{1-\phi^2}.$$

- Putting all facts together gives us

$$\sqrt{T}\left(\widehat{\phi}_T - \phi\right) \xrightarrow{d} N(0, 1-\phi^2).$$

Remark

Observe that for $\phi = 1$ the variance is zero!

Estimation of the AR model

Remark

The last fact is key in the analysis of the AR models. It allows us to run tests and construct confidence intervals for ϕ. The 95% confidence interval will be

$$\left[\hat{\phi}_T - 1.96\sqrt{\frac{1-\phi^2}{T}}, \hat{\phi}_T + 1.96\sqrt{\frac{1-\phi^2}{T}}\right].$$

Estimation of the MA model

• There are several different ways to estimate MA models. For simplicity we consider the basic MA(1) model

$$y_t = \varepsilon_t + \theta\varepsilon_{t-1},$$

where $|\theta| < 1$ and $\varepsilon_t \sim$iid$(0, \sigma^2)$.

[Approach 1] Using ACF, i.e.,

$$\rho_1 = \frac{\theta}{1 + \theta^2}$$

We can use this expression to estimate θ since we know how to estimate ρ_1

$$\widehat{\rho}_1 = \frac{\sum_{t=2}^{T}(y_t - \overline{y}_T)(y_{t-1} - \overline{y}_T)}{\sum_{t=2}^{T}(y_t - \overline{y}_T)^2}$$

Estimation of the MA model

Then we will have to solve the following second order equation:

$$\widehat{\rho}_1 \theta^2 - \theta + \widehat{\rho}_1 = 0$$

This gives us the following possible solutions

$$\widehat{\theta} = \begin{cases} \frac{1-\sqrt{1-4\widehat{\rho}_1^2}}{2\widehat{\rho}_1} \text{ if } 0 < |\widehat{\rho}_1| < 0.5 \\ \pm 1 \text{ if } |\widehat{\rho}_1| = 0.5 \\ \text{does not exist if } |\widehat{\rho}_1| > 0.5 \\ 0 \text{ if } \widehat{\rho}_1 = 0 \end{cases}$$

Estimation of the MA model

[Approach 2] The other approach is based on the least square principle

$$\min_{\theta} \sum_{t=2}^{T} \varepsilon_t^2 = \min_{\theta} \sum_{t=2}^{T} (y_t - \theta\varepsilon_{t-1})^2$$

This is nonlinear optimization problem. (Write down the first order conditions and compare them with those of an AR(1))

We can show that

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \xrightarrow{d} N(0, (1-\theta)^2).$$

Estimation of the MA model

[Approach 3] Hannan-Rissanem estimator:

Step1 Estimate high order AR and obtain errors

$$\widehat{\varepsilon}_t = y_t - \mathbb{E}[y_t | y_{t-1}, y_{t-2}, ...]$$

Step2 Regress y_t on estimated errors using OLS:

$$y_t = \theta \widehat{\varepsilon}_{t-1} + \varepsilon_t$$

and obtain

$$\widehat{\theta} = \frac{\sum_{t=2}^{T} y_t \widehat{\varepsilon}_{t-1}}{\sum_{t=2}^{T} \widehat{\varepsilon}_{t-1}^2}$$