Estimation and Inference. Part I

Short overview:

(1) Short Summary

(2) Estimation of time series models

- Estimation of the mean and autocovariances
- Estimation of the AR model
- Estimation of the MA model

Short Summary

So far, we have met and analyzed the parametric family of stochastic processes defined as ARMA(p,q). Either for the general case or for specific processes, we have seen:

- how to retrieve the autocovariances, the ACF, and the PACF;
- what are the conditions for stationarity, causality, and invertibility;
- how to express weakly stationary processes as a linear combination of a sequence of uncorrelated random variables (Wold's decomposition).

What's next?

- At this point, we are ready to go back to economic data. In the real world, we do not know the true DGP, and the only source of information is the observed time series $\{y_t\}_1^T$ (i.e., a unique realization of the DGP).

- On the basis of the known properties and characteristics of general ARMA(p,q) processes, we can model our data and make inference on the characteristics of the true DGP.

- We'll continue with the following steps:
  - identification (model selection);
  - estimation;
  - diagnostic checks.

(2) Estimation of time series models

• Consider ARMA(p,q) model

$$\begin{aligned} y_t &= \mu + \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q} \\ &= \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \end{aligned}$$

• For now we will assume that the orders p and q for ARMA(p,q) are known. (Methods for selection of p and q will be discussed after this section)

• We can (have to) estimate unknown parameters of the model:

- mean of y_t;
- variance and autocovariances
- parameters $\phi_1, \phi_2, ..., \phi_p$ and $\theta_1, \theta_2, ..., \theta_q$

Estimation of the mean

- An estimator for the mean of a process $\{y_t\}$ can be formulated as

$$\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$$

- We need to check the consistency of the estimator and its asymptotic distribution

Remark (Law of Large Numbers)

Given some restrictions on the dependence and heterogeneity and moments of a sequence of random variables $\{z_t\}$ *we have*

$$\bar{z}_T - \mathbb{E}[z_T] \xrightarrow{p} 0 \text{ as } T \to \infty, \tag{1}$$

where $\bar{z}_T = \frac{1}{T}\sum_{t=1}^{T} z_t$*. Equation* (1) *is called Law of Large Numbers.*

Estimation of the mean

Definition

A process $\{x_t\}$ is called a martingale difference sequence (mds) if

$$\mathbb{E}[x_t | x_{t-1}, x_{t-2}, ...] = 0.$$

Theorem (Law of Large Numbers)

If $y_t = \mu + \psi(L)\varepsilon_t$, *where* $\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + ...$,

(i) $\sum_{j=1}^{\infty} (j\psi_j)^2 < \infty$

(ii) ε_t *is mds, strictly stationary and ergodic with* $Var(\varepsilon_t) = \sigma^2 < \infty$.

then we have

$$\frac{1}{T} \sum_{t=1}^{T} y_t \xrightarrow{p} \mathbb{E}[y_t] = \mu,$$

as $T \to \infty$.

Example

- Assume we have stationary AR(1) $y_t = \mu + \phi y_{t-1} + \varepsilon_t$ and ε_t is $WN(0, \sigma^2)$.

- We know that $\mathbb{E}[y_t] = \frac{\mu}{1-\phi}$ for all T.

- Then from the LLN we have that

$$\frac{1}{T}\sum_{t=1}^{T} y_t \xrightarrow{p} \mathbb{E}[y_t] = \frac{\mu}{1-\phi}$$

Remark (Central Limit Theorem)

Given some restrictions on the dependence and heterogeneity and moments of a sequence of random variables $\{z_t\}$ we have

$$\sqrt{T}(\bar{z}_T - \mathbb{E}[z_T]) \xrightarrow{d} N(0, \sigma^2) \text{ as } T \to \infty \tag{2}$$

where $\sigma^2 = \lim_{T\to\infty} T\ Var(\bar{z}_T)$. Equation (2) is called Central Limit Theorem.

Estimation of the mean

Theorem (Central Limit Theorem)

If $y_t = \mu + \psi(L)\varepsilon_t$*, where* $\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + ...,$

(i) $\sum_{j=1}^{\infty} |j||\psi_j| < \infty$

(ii) ε_t *is mds, strictly stationary and ergodic with* $Var(\varepsilon_t) = \sigma^2 < \infty$

then we have

$$\sqrt{T}\left(\bar{y}_T - \mathbb{E}[y_t]\right) = \sqrt{T}\left(\frac{\sum_{t=1}^{T} y_t}{T} - \mu\right) \xrightarrow{d} N(0, \psi^2(1)\sigma^2),$$

as $T \rightarrow \infty$*.*

Remark

Term $\psi^2(1)\sigma^2$ *is called long-run variance.*

Remark (General Result)

If an estimator $\widehat{\theta}_T$ of some parameter θ satisfies a central limit theorem

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \xrightarrow{d} N(0, V),$$

then the confidence interval can be written as

$$\widehat{\theta}_T \pm Z_{1-\alpha/2}\sqrt{\frac{V}{T}}.$$

(for 95% conf. interval we have $Z_{1-\alpha/2} = 1.96$)

Remark (General time series process)

For process $y_t = \mu + \psi(L)\varepsilon_t$, where $\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + ...$, under some regularity conditions we have

$$\sqrt{T}\,(\overline{y}_T - \mathbb{E}[y_t]) = \sqrt{T}\left(\frac{\sum_{t=1}^{T} y_t}{T} - \mu\right) \xrightarrow{d} N(0, \psi^2(1)\sigma^2).$$

Then the confidence interval can be written as

$$\overline{y}_T \pm Z_{1-\alpha/2}\sqrt{\frac{\psi^2(1)\sigma^2}{T}}.$$

(for 95% conf. interval we have $Z_{1-\alpha/2} = 1.96$)

Example

- Lets look again on the stationary AR(1)

$$y_t = \phi y_{t-1} + \varepsilon_t.$$

- Then $\psi(L) = \frac{1}{1-\phi L}$ and $\psi(1) = \frac{1}{1-\phi}$.
- Thus from CLT (previous slide)

$$\sqrt{T}\,\overline{y}_T \xrightarrow{d} N\left(0, \frac{\sigma^2}{(1-\phi)^2}\right),$$

• With above results we can test and build confidence intervals for the mean of an AR(1) process.

Remark

The 95% confidence interval for the estimator of the mean is given by

$$\left[\overline{y}_T - 1.96\sqrt{\frac{\sigma^2}{T(1-\phi)^2}}, \overline{y}_T + 1.96\sqrt{\frac{\sigma^2}{T(1-\phi)^2}}\right].$$

This fact follows from CLT.

• Observe the problem that occurs with $\phi = 1$.