Chapter 1:

Characteristics of economic time series data

Short overview:

(1) What is time series? What is Stochastic processes
(2) Important steps for the Econometric Analsis
(3) Descriptive statistics
(4) Autocorrelation function
(5) Partial autocorrelation function
(6) Stationarity
(7) Ergodicity

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Time series and economic data

Definition: A time series is a set of observations x_t observed in sequence over time, $t = 1, ..., T$.

To indicate the dependence on time, we adopt new notation, and use the subscript t to denote the individual observation, and T to denote the number of observations.

Because of the sequential nature of time series, we expect that x_t and x_{t-1} are not independent, so classical assumptions are not valid:

The past can affect the future, but not vice versa.

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We can separate time series into two categories:

1) **Univariate** where $x_t \in \mathbb{R}$ is scalar

Example: GDP_t =Gross Domestic Product at time t

2) **Multivariate** where $x_t \in \mathbb{R}^m$ is vector-valued

Example:

$$\begin{pmatrix} GDP_t \\ r_t \\ P_t \\ M_t \end{pmatrix}$$

GDP_t = Gross Domestic Product at time t, r_t = Interest rate at time t, P_t =Price level at time t, M_t =Money at time t.

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We distinguish between discrete and continuos time series:

If t are discrete $\Rightarrow$ discrete time series

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If t are continuos $\Rightarrow$ continuous time series

Example: Stock price

In this course we focus on the discrete time series.

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Stochastic process

Definition: A stochastic process is a collection of random variables such that to each element $t \in T_0$ is associated a random variable Y_t.

$\rightarrow$ The process can be written $\{Y_t : t \in T_0\}$.

If $T_0 = \mathbb{R}$ (real numbers) we have a process in continuous time.

If $T_0 = Z$ (integers) or $T_0 = Z$ we have a discrete time process.

$\rightarrow$ To observe a time series is equivalent to observing a realization of a process $\{Y_t : t \in T_0\}$ or a portion of such a realization

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Important steps for the Econometric Analysis

Suppose we have time series data (one or several economic variables) and we wish to study its evolution in time or/and to understand the relationship between the variables (how one variable affect another one). We have to follow the following main steps:

1) Descriptive statistics, graphical analysis and visual inspection of the data $\Longrightarrow$ Mean, variance, covariance,..., Plot (figures) and analyze visually the data: stationary & non stationary, seasonality,...

2) Analyze the properties of the data: Autocorrelation and partial autocorrelation functions

3) Model selection and Estimation

4) Validation

5) Forecasting.

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Descriptive statistics

- Mean:

$$E(Y_t) = \mu_t = \int Y_t f(y_t) dy_t$$

- Variance:

$$Var(Y_t) = E[(Y_t - E(Y_t))^2] = E(Y_t^2) - E(Y_t)^2$$

- Standard deviation:

$$\sigma_X = \sqrt{Var(\cdot)}$$

- Covariance:

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

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$$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \, \sigma_Y} \in [-1, 1]$$

- $Cov(X, Y) = 0 \Leftrightarrow E(XY) = E(X)E(Y) \Leftrightarrow$ no correlation between X and Y.

- X and Y independent $\Rightarrow Cov(X, Y) = 0$, but not vice versa

- Useful properties:

(1) Expectation is linear: $E(aX + bY) = aE(X) + bE(Y)$

(2) Variance is **NOT** linear: $Var(aX + b) = a^2 Var(X)$

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Autocorrelation Function (ACF)

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$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

- **Autocovariance:**
The autocovariance function of a stochastic process Y_t is a covariance between two elements of the series, i.e.,

$$\gamma_{t_1,t_2} = cov(Y_{t_1}, Y_{t_2}),$$

is the autocovariance between element t_1 and t_2. If $t_1 = t_2 = t$, then the autocovariance function is equal to the variance.

$$\gamma_{t_1,t_2} = \sigma_t^2$$

Variances and autocovariances are all expressed in terms of the squared unit of measure of Y_t.

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For $t_1 = t_2 = t \Longrightarrow \rho_{t_1,t_2} = 1$.

- ACF is a (crucial) starting point to describe time dependencies in a stochastic process.

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The Partial Autocorrelation Function (PACF)

An important part of the correlation between Y_t and Y_{t-k} may arise from their correlation with the intermediate variables $Y_{t-1}, ..., Y_{t-k+1}$. To control for this, we define the Partial Autocorrelation Function P_k (PACF):

$$P_k = Corr(Y_t, Y_{t-k} | Y_{t-1}, ..., Y_{t-k+1}).$$

The PACF varies between -1 and 1 (like ACF), with values near $\pm$ indicating stronger correlation. The PACF filters out the effect of "shorter" lags autocorrelation from the correlation at "longer" lags.

The Partial Autocorrelation Function (PACF)

- **How to calculate PACF?** We can obtain from Yule-Walker equations:

$$P_k = \frac{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \cdots & \rho_{k-3} & \rho_2 \\ \cdots & \cdots & & \cdots & \cdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \cdots & \cdots & & \cdots & \cdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & 1 \end{vmatrix}}.$$

- **Examples for P_1, P_2 and P_3?**

Stationarity and Ergodicity

Motivation for additional assumptions

- A stochastic process can be described by *n*-dimentional probability distributions. In particular,

Definition (Distribution of a stochastic process)

A distribution function of a stochastic process $\{Y_t : t \in T_0\}$ can be defined by specifying, for each subset $t_1, ..., t_n \in T$ with $n \geq 1$, the joint distibution function of $(Y_{t_1}, ..., Y_{t_n})$, i.e.,

$$F(y_1, ..., y_n; t_1, ..., t_n) = P[Y_{t_1} \leq y_1, ..., Y_{t_n} \leq y_n].$$

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$$F(y_1, ..., y_n; t_1, ..., t_n) = P[Y_{t_1} \leq y_1, ..., Y_{t_n} \leq y_n].$$

Motivation for additional assumptions

Remark

Specifying the complete shape of the distribution is too ambitious. Why? Consider only first and second moments of a stochastic process:

- $\mathbb{E}(Y_t) = \mu_t$ *for each* $t = 1, ..., T$. *This gives* T *values.*
- $\mathbb{E}(Y_t - \mu_t)^2 = \sigma_t^2$ *for each* $t = 1, ..., T$. *Also gives* T *values.*
- $\mathbb{E}[(Y_{t_1} - \mu_{t_1})(Y_{t_2} - \mu_{t_2})] = \gamma_{t_1,t_2}$ *for each* $t_1, t_2 = 1, ..., T$, *which gives* $\frac{T(T-1)}{2}$.

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Motivation for additional assumptions

Conclusion

Therefore we need to make additional (simplifying) assumptions:

(1) *Stationarity*

(2) *Ergodicity*

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(2) Ergodicity

Stationarity

There are two concepts of stationarity of stochastic processes: Strict and weak stationarity.

Definition (Strict stationarity)

A process is said to be strictly stationary if for any values of $(s_1, s_2, ..., s_n)$ the joint distribution of $(Y_{t+s_1}, ..., Y_{t+s_n})$ depends only on the intervals separating the dates $s_1, s_2, ..., s_n$ and not on the date itself (t).

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Stationarity

Definition (Weak stationarity)

The process Y_t is said to be weakly-stationary or covariance-stationary if

- $\mathbb{E}(Y_t) = \mu$ for all t;
- $\mathbb{E}[(Y_{t_1} - \mu)(Y_{t-j} - \mu)] = \gamma_j$ for all t and any j.

Examples of stationary stochastic processes

One of the most basic processes:

Example (White Noise)

A sequence of random variables $\{\varepsilon_t\}$ is called a white noise if the following holds

$\mathbb{E}(\varepsilon_t) = 0$ for all t;

$\mathbb{E}(\varepsilon_t^2) = \sigma^2$ for all t;

$\mathbb{E}(\varepsilon_t \varepsilon_s) = 0$ for all $t \neq s$.

Example (Example 2)

We have a process:

$$Z_t = \begin{cases} Y_t, \text{ if } t \text{ is odd,} \\ Y_t + 1, \text{ if } t \text{ is even,} \end{cases}$$

where Y_t is a stationary series. Is Z_t weakly stationary?

Example (Example 3)

Define the process

$$S_t = Y_1 + ... + Y_t,$$

where Y_t is iid $(0, \sigma^2)$. Show that for $h > 0$

$$Cov(S_{t+h}, S_t) = t\sigma^2$$

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Ergodicity

In order to do the empirical analysis with time series observations stationarity assumption is not enough. Why?

Until now, we only defined theoretical moments(population moments) of a stochastic process. However these moments are unknown in practice and we need to estimate them from a single observed realization $\{Y_t\}_{t=1}^{T}$ of a stochastic process.

In order to make use the population moments we need to estimate them and for that we need to assume additionally ergodicity.

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- Informally speaking, a stochastic process $\{Y_t\}$ is ergodic if any two collections of random variables partitioned far apart in the sequence are almost independently distributed.
- The formal definition is a bit technical:

Definition (Ergodicity)

A stationary stochastic process $\{Y_t\}$ is called ergodic if for any t, k, l and any bounded functions g and h

$$\lim_{T\to\infty} Cov\,(g(Y_t, ...Y_{t+k}), h(Y_{t+k+T}, ..., Y_{t+k+T+l})) = 0.$$

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Example

Consider a process $Y_t = Z + U_t$, where $\{U_t\}$ are iid$[0,1]$ and Z is random variable distributed as $N(0,1)$. Z and U_t are independent. Is Y_t weakly stationary? Is it ergodic for the mean?

One of the most important implications is **consistency** of the sample estimators:

- Sample mean: $\overline{Y}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t$ is an estimator of $\mathbb{E}[Y_t]$.
- Sample Covariance: $\widehat{\gamma}_k = \frac{1}{T}\sum_{t=1}^{T}(Y_t - \overline{Y}_T)(Y_{t-k} - \overline{Y}_T)$ is an estimator of $Cov(Y_t, Y_{t-1})$
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Theorem (Law of Large Numbers,LLN)

Let $\{Y_t\}$ be a stationary and ergodic stochastic process. Then

$$\overline{Y}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t \xrightarrow{p} \mathbb{E}[Y_t].$$

Theorem

If Y_t is strictly stationary and ergodic and $\mathbb{E}(Y_t^2) < \infty$, then as $T \to \infty$,

(1) $\widehat{\gamma}_k \xrightarrow{p} \gamma_k$;

(2) $\widehat{\rho}_k \xrightarrow{p} \rho_k$.

Discussion: Sufficient conditions

- LLN tells us that $\overline{Y}_T$ is a consistent estimator of $\mathbb{E}[Y_t]$.
- Recall from Econometrics I: Sufficient conditions for the consistency of an estimator $\widehat{\theta}_T$ are

$$\lim_{T\to\infty} \mathbb{E}(\widehat{\theta}_T) = \theta, \text{ and } \lim_{T\to\infty} Var(\widehat{\theta}_T) = 0. \qquad (1)$$

- Then we can also derive a sufficient condition for LLN (or ergodicity)

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Finally, when do we have

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Discussion: Sufficient condition for ergodicity

Remark

A sufficient condition for ergodicity for the mean

$$\sum_k |\gamma_k| < \infty.$$