Simultaneous Equation Approach to Cointegration: Estimation and Inference

Everything started as follows:

$$\left[\begin{array}{l} y_t = \beta x_t + z_t \; ; \quad z_t = \rho z_{t-1} + e_{zt} \; ; \quad |\rho| < 1 \\ \Delta x_t = e_{xt} \end{array}\right.$$

$$\left[\begin{array}{l} \Delta y_t = \beta \Delta x_t + \Delta z_t \\ \Delta x_t = e_{xt} \end{array}\right. \; ; \quad \begin{array}{l} \Delta y_t = \beta e_{xt} + (\rho - 1) z_{t-1} + e_{zt} \\ \Delta x_t = e_{xt} \end{array}$$

$$\left[\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = \begin{pmatrix} \rho - 1 \\ 0 \end{pmatrix} (1, -\beta) \begin{pmatrix} y_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}\right] \quad \text{with } \begin{array}{l} u_{1t} = \beta e_{xt} + e_{zt} \\ u_{2t} = e_{xt} \end{array}$$

$$\boxed{\Delta X_t = \underset{2 \times 2}{\Pi} X_{t-1} + u_t} \qquad X = \begin{pmatrix} y \\ x \end{pmatrix}$$

Testing for cointegration is testing for the rank of Π. Three situations:

* $r(\Pi) = 2$ I(0) and no cointegration
* $r(\Pi) = 1$ I(1) and cointegration
+ $r(\Pi) = 0$ I(1) and <u>no</u> cointegration

At the same time we test for the rank of Π we want to estimate the coint. vector $(1, -\beta)$. In order to do both things simultaneously we will be using REDUCED RANK REGRESSION TECHNIQUES.

In a general set-up (no deterministic components yet):

$$\underset{p\times 1}{\Delta X_t} = \underset{p\times r}{\alpha}\,\underset{r\times p}{\beta'}\,\underset{p\times 1}{X_{t-1}} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t \qquad t=1,\dots T$$

$$\varepsilon_t \sim \text{Normal}(0, \Omega)$$

- The number of lags has been selected by some information criteria in a VAR in levels (Why? Because if there is cointegration it DOES NOT EXIST a VAR in Δ)

$$AIC = \ln|\hat{\Omega}| + p^2 k \frac{2}{T}$$

$$SC = \ln|\hat{\Omega}| + p^2 k \frac{\ln T}{T}$$

$$H\text{-}Q = \ln|\hat{\Omega}| + p^2 k \frac{2\ln\ln T}{T}$$

To focus on the parameters of interest r, α, β, Ω

Regress ΔX_t on $\Delta X_{t-1}, \dots, \Delta X_{t-k+1} \longrightarrow \boxed{R_{0t}}$

Regress X_{t-1} on $\Delta X_{t-1}, \dots, \Delta X_{t-k+1} \longrightarrow \boxed{R_{1t}}$

$$\log L(\alpha, \beta, \Omega) = -\frac{1}{2} T \log|\Omega| - \frac{1}{2}\sum_{t=1}^{T} (R_{0t} - \alpha\beta' R_{1t})' \Omega^{-1} (R_{0t} - \alpha\beta' R_{1t})$$

Define $\boxed{S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it} R_{jt}'} \qquad i,j = 0,1$

Note that for fixed β

$$\hat{\alpha}(\beta) = S_{01}\beta(\beta' S_{11}\beta)^{-1} \quad \text{and}$$

$$\hat{\Omega}(\beta) = S_{00} - S_{01}\beta(\beta' S_{11}\beta)^{-1}\beta' S_{10}$$
$$= S_{00} - \hat{\alpha}(\beta)(\beta' S_{11}\beta)\hat{\alpha}(\beta)'$$

From "The cointegrated VAR Model" by Katarina Juselius (2006, Oxford press)

$$L_{max}^{-\frac{2}{T}}(\hat{\alpha}(\beta), \beta, \hat{\Omega}(\beta)) = L_{max}^{-\frac{2}{T}}(\beta) = |\hat{\Omega}(\beta)| + \text{cte term}$$

This is due to the Normality assumption

$$\hat{\Omega}(\beta, \alpha) = T^{-1} \sum (R_{0t} - \alpha\beta' R_{1t})(R_{0t} - \alpha\beta' R_{1t})'$$

$$= T^{-1}\left(\sum R_{0} R_{0t}' - \sum R_{0t} R_{1t}' \beta\alpha' - \alpha\beta' \sum R_{1t} R_{0t}' + \alpha\beta' \sum R_{1t} R_{1t}' \beta\alpha'\right)$$

$$= S_{00} - S_{01}\beta\alpha' - \alpha\beta' S_{10} + \alpha\beta' S_{11}\beta\alpha'$$

$$\hat{\alpha}(\beta) = S_{01}\beta(\beta' S_{11}\beta)^{-1}$$

$$\hat{\Omega}(\beta) = S_{00} - S_{01}\beta(\beta' S_{11}\beta)^{-1}\beta' S_{10}$$

The ML estimator of β is given by $\hat{\beta}$ that minimize $|\hat{\Omega}(\beta)|$. To derive this estimator we use the following result

$$\begin{vmatrix} A & B \\ B' & C \end{vmatrix} = |A|\,|C - B'A^{-1}B| = |C|\,|A - BC^{-1}B'|$$

Now substitute $\quad S_{00} = A$

$\beta' S_{11}\beta = C$

$S_{01}\beta = B$

$$|S_{00}|\,|\beta' S_{11}\beta - \beta' S_{10} S_{00}^{-1} S_{01}\beta| = |\beta' S_{11}\beta|\,\underbrace{|S_{00} - S_{01}\beta(\beta' S_{11}\beta)^{-1}\beta' S_{10}|}_{|\Omega(\beta)|}$$

Hence

$$|\hat{\Omega}(\beta)| = \frac{|S_{00}|\,|\beta' S_{11}\beta - \beta' S_{10} S_{00}^{-1} S_{01}\beta|}{|\beta' S_{11}\beta|}$$

$$= |S_{00}|\,\frac{|\beta'(S_{11} - S_{10} S_{00}^{-1} S_{01})\beta|}{|\beta' S_{11}\beta|}$$

Using the result that

$$f(x) = \frac{|x'Mx|}{|x'Nx|}$$

is maximized by solving the eigenvalue problem

$$|\rho N - M| = 0$$

we can obtain a solution for β that min $|\hat{\Omega}(\beta)|$

Take

$$M = S_{11} - S_{10} S_{00}^{-1} S_{01}$$
$$N = S_{11}$$
$$x = \beta$$

The eigenvalue problem that gives the estimate of β is

$$|\rho S_{11} - S_{11} + S_{10} S_{00}^{-1} S_{01}| = 0$$

or equivalently

$$|\underbrace{(1-\rho)}_{\lambda} S_{11} - S_{10} S_{00}^{-1} S_{01}| = 0$$

From this eigenvalue problem we get

$$\hat{\lambda}_1 \geq \dots > \hat{\lambda}_p \quad \text{eigenvalue}$$

$$\hat{v}_1 \quad \hat{v}_p \quad \text{eigenvectors}$$

$$|\hat{\Omega}(\beta)| = |S_{00}| \prod_{i=1}^{p} (1 - \lambda_i)$$

The magnitude of $\hat{\lambda}_i$ is a measure of the "stationarity" of the corresponding $v_i' x_t$, the larger the $\hat{\lambda}_i$, the "more" stationary is the relation.

Think on terms of canonical correlations.

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The Likelihood ratio test of rank "r" vs p is

$$Q\left(H(r) \mid H(p)\right)^{-\frac{2}{T}} = \frac{|S_{00}|}{|S_{00}|} \frac{\prod_{i=1}^{r}(1-\hat\lambda_i)}{\prod_{i=1}^{p}(1-\hat\lambda_i)}$$ so

$$\boxed{-2\log Q\left(H(r) \mid H(p)\right) = -T \sum_{i=r+1}^{p} \log(1-\hat\lambda_i)}$$

Example: p = 3 how to proceed?

$H_0: r \le 2 \qquad H_A \; r=3 \qquad -T\log(1-\hat\lambda_3)$

$H_0: r \le 1 \qquad H_A \; r=3 \qquad -T\left(\log(1-\hat\lambda_2)+\log(1-\hat\lambda_3)\right)$

$H_0: r = 0 \qquad H_A \; r=3 \qquad -T\left(\log(1-\hat\lambda_1)+\log(1-\hat\lambda_2)+\log(1-\hat\lambda_3)\right)$

Theorem: Under the hypothesis

$$H(r): \Pi = \alpha\beta'$$

the ML estimator of β is found by the following procedure: (1) solve

$$|\lambda S_{11} - S_{10} S_{00}^{-1} S_{01}| = 0$$

for the eigenvalues $1 > \hat\lambda_1 > \ldots > \hat\lambda_p > 0$

and eigenvectors

$$\hat V = (\hat v_1, \ldots \hat v_p)$$

which normalizes $\hat V' S_{11} \hat V = I$.

The cointe relations are estimated by

$$\hat\beta = (\hat v_1, \ldots, \hat v_r)$$

and

$$L_{max}^{-\frac{2}{T}}(H(r)) = |S_{00}| \prod_{i=1}^{r}(1-\hat\lambda_i)$$

Theorem: The AD of the LRT for the hypothesis $\underset{p\times r}{\Pi} = \alpha \underset{r \times p}{\beta'}$ is in general given by the DF distribution with p-r degrees of freedom

$$\operatorname{tr}\left\{ \int_0^1 (dB) F' \left[\int_0^1 F F' du \right]^{-1} \int_0^1 F (dB)' \right.$$

$\to B = BM_{p-r}$ $\qquad \to F = f(B)$

(see Johansen "Likelihood-based inference in cointegrated vector autoregressive models" (1995, Oxford U. press))
pg 156
Thm 11.1

Theorem (13.3 Johansen's book)

- $\hat{\beta}$ normalized $\overset{A.D}{\sim}$ mixed Gaussian

  (Hypothesis tests on $\beta \sim \chi^2$)

- $\sqrt{T}(\hat{\alpha} - \alpha) \sim N(0, V)$

Testing restrictions on $\underset{\sim}{\beta}$

$$\underset{p\times r}{\beta^c} = \underset{p\times s}{H}\underset{s\times r}{\varphi}$$

Examples:

- $H\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ 0 \\ 0 \end{pmatrix}$

- • $H\varphi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix}$

- • • $H = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ what are we testing?

$H^c(r) = \beta^c = H\varphi$ or $H^c(r) : R'\beta = 0$

with $H = R_\perp$

Under $H^c(r)$: $\Delta X_t = \alpha \varphi' H' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t$

We can find the restricted ML estimator by solving reduced-rank regression of ΔX_t on $H' X_{t-1}$ corrected by the short-run

$$|\lambda H' S_{11} H - H' S_{10} S_{00}^{-1} S_{01} H| = 0$$

This gives "s" eigenvalues $\lambda_1^c, \dots \lambda_s^c$ and the eigenvectors $v_1^c, \dots v_s^c$.

Theorem (7.2 Johansen)

$$-2 \log Q(H_0 | H^c(r)) = T \sum_{i=1}^{r} \log\{(1-\lambda_i^c)/(1-\hat{\lambda}_i)\} \sim \chi^2_{r(p-s)}$$

Dealing with Deterministic Components

$$\Delta X_t = \alpha\beta' X_{t-1} + \underline{\underline{\mu_0 + \mu_1 t}} + \varepsilon_t$$

It could be that

$$\mu_0 = \alpha\beta_0 + \gamma_0$$

$$\mu_1 = \alpha\beta_1 + \gamma_1$$

So $\Delta X_t = \alpha\beta' X_{t-1} + \alpha\beta_0 + \alpha\beta_1 t + \gamma_0 + \gamma_1 t + \varepsilon_t$

or

$$\Delta X_t = \alpha\left[\beta', \beta_0, \beta_1\right]\begin{bmatrix} X_{t-1} \\ 1 \\ t \end{bmatrix} + \gamma_0 + \gamma_1 t + \varepsilon_t$$

__Case I__: $\mu_1 = \mu_0 = 0$ $\left(E(\Delta X_t) = 0,\ E(\beta' X_t) = 0\right)$

__Case II__: $\mu_1 = 0$, $\gamma_0 = 0$ but $\beta_0 \neq 0$ ($\Rightarrow E(\Delta X_t) = 0$)
The only deterministic component is the intercept in the CI relationship

__Case III__: $\mu_1 = 0$ (i.e. $\beta_1 = \gamma_1 = 0$)
$\gamma_0 \neq 0$

__Case IV__: $\gamma_1 = 0$ but $(\gamma_0, \beta_0, \beta_1) \neq 0$
The trend appears in the C-I relationship

__Case V__: No restrictions on μ_0, μ_1

AND FROM HERE TO GONZALO-GRANGER
P-T DECOMPOSITION.