HAC Standard Errors

$$y_t = \beta_0 + \beta_1 X_t + u_t$$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_1^T (X_t - \bar{X}) u_t}{\sum_1^T (X_t - \bar{X})^2}$$

$$\sqrt{T}\left(\hat{\beta}_1 - \beta_1\right) = \frac{\frac{1}{\sqrt{T}} \sum_1^T (X_t - \bar{X}) u_t}{\frac{1}{T} \sum_1^T (X_t - \bar{X})^2}$$

$$\frac{1}{T} \sum_1^T (X_t - \bar{X})^2 \longrightarrow \sigma_X^2$$

$$\boxed{V_t = (X_t - \bar{X}) u_t} = x_t u_t \quad \text{with} \quad x_t = X_t - \bar{X}$$

The crucial aspect of the A.D of $\hat{\beta}_1$ will depend on

$$Var\left(\frac{1}{\sqrt{T}} \sum_1^T V_t\right) =$$

$$\frac{1}{T} Var\left(\sum_1^T V_t\right) = \frac{1}{T}\left[\begin{array}{l} Var(V_1) + Cov(V_1 V_2) + \dots + Cov(V_1 V_T) \\ Cov(V_2 V_1) + V(V_2) + \dots + Cov(V_2 V_T) \\ - - - - \\ Cov(V_T V_1) + \dots + V(V_T) \end{array}\right]$$

Several scenarios

(I) $Cov(V_t V_{t-j}) = 0 \quad \forall j$ and $E(u_t^2 | X_t) = \sigma_t^2$

$$Var\left(\frac{1}{\sqrt{T}} \sum_1^T V_t\right) = \frac{1}{T} \sum_1^T E(x_t u_t)^2 = \frac{1}{T}\left[E(x_1 u_1)^2 + \dots + E(x_T u_T)^2\right]$$

How do we estimate $Var\left(\frac{1}{\sqrt{T}} \sum_1^T V_t\right)$ consistently?

$$\frac{1}{T} \sum_1^T x_t^2 \underset{\uparrow \text{OLS}}{\hat{u}_t^2} - \frac{1}{T} \sum_1^T x_t^2 u_t^2 \xrightarrow{P} 0$$

$$\frac{1}{T} \sum_1^T x_t^2 u_t^2 \xrightarrow{P} \frac{1}{T} \sum_1^T E(x_t^2 u_t^2)$$

$$Var\left(\sqrt{T}(\hat{\beta}_T - \beta)\right) = \frac{\frac{1}{T} \sum_1^T x_t^2 \hat{u}_t^2}{(\hat{\sigma}_x^2)^2}$$

$$\boxed{Var(\hat{\beta}_T) = \frac{\sum_1^T x_t^2 \hat{u}_t^2}{T^2 (\hat{\sigma}_x^2)^2}}$$

Check that you recover the standard expression when $E(u_t^2 | X_t) = \underline{\sigma^2}$.

(D) Suppose $Cov(v_t v_{t-k}) = 0$ for $k > 1$ (for instance)

$Var\left(\frac{1}{\sqrt{T}} \sum_1^T v_t\right)$

$= \frac{1}{T}\left[\sum_1^T E(x_t u_t)^2 + 2 \sum_2^T E(x_t u_t x_{t-1} u_{t-1})\right]$

$= \frac{1}{T} \sum_1^T E(x_t u_t)^2 + 2 \frac{T-1}{T} \gamma_1$ with $\gamma_1 = cov(v_t v_{t-1})$

A consistent estimator of this variance is

$\frac{1}{T} \sum_1^T (x_t \hat{u}_t)^2 + \frac{2}{T} \sum_2^T (x_t \hat{u}_t \hat{u}_{t-1} x_{t-1})$

This can be simplify if we assume

$E(x_t u_t)^2 = Var(v_t) = \sigma_v^2$

$\frac{T}{T}\sigma_v^2 + 2 \frac{T-1}{T} \gamma_1 = \sigma_v^2 \left(1 + 2 \frac{T-1}{T} \frac{\gamma_1}{\sigma_v^2}\right)$

$= \boxed{\sigma_v^2 \left(1 + 2 \frac{T-1}{T} \rho_1\right)}$

$= \sigma_v^2 f_T$

Estimated consistently by

$\boxed{\hat{\sigma}_v^2 \left(1 + 2 \frac{T-1}{T} \hat{\rho}_1\right)}$

(IV) Suppose $Cov(v_t v_{t-\tau}) \to 0$ as $\tau \to \infty$

$$Var\left(\frac{1}{\sqrt{T}} \sum v_t\right)$$

$$= \frac{1}{T}\left[\sum_1^T E(x_t u_t)^2 + 2\sum_2^T E(x_t u_t u_{t-1} x_{t-1}) + \dots + 2\, E(x_t u_t u_{t-m} x_{t-m})\right]$$

$m = T-1$

We could think that a consistent estimator will be

$$\frac{1}{T}\sum(x_t \hat{u}_t)^2 + \frac{2}{T}\sum_2^T(x_t \hat{u}_t \hat{u}_{t-1} x_{t-1}) + \dots + \frac{2}{T}\sum_m^T(x_t \hat{u}_t \hat{u}_{t-m} x_{t-m})$$

$m = T-1$

only one observation

The answer is **not** (discussed in class).

It can even happen that this becomes **negative**.

A consistent estimator is

$$\frac{1}{T}\sum(x_t \hat{u}_t)^2 + 2\sum_{j=1}^{\tau} w(j,\tau)\, \boxed{\frac{1}{T}\sum_{j+1}^{T}(x_t \hat{u}_t \hat{u}_{t-j} x_{t-j})}$$

with $w(j,\tau) = 1 - [j/(\tau+1)]$

and $\tau(T) \approx T^{\frac{1}{3}}$

This is the approach by Newey-West (Econometrica 1987)

This expression can be simplified if we assume homoscedasticity:

$$\frac{1}{T}\sum_{1}^{T} E(X_t u_t)^2 = \underline{\underline{\sigma_v^2}}$$

Then

$$Var\left(\frac{1}{\sqrt{T}}\sum_{1}^{T} v_t\right) = \sigma_v^2 f_T$$

with $f_{\cancel{T}} = 1 + 2\sum_{j=1}^{T-1}\left(\frac{T-j}{T}\right)\rho_j$

$$Var(\hat{\beta}_1) = \boxed{\frac{1}{T}\frac{\sigma_v^2}{(\sigma_x^2)^2}} f_T$$

$$\boxed{\begin{aligned} \tilde{\sigma}_{\hat{\beta}}^2 &= \hat{\sigma}_{\hat{\beta}}^2 \hat{f}_T \\ \hat{f}_t &= 1 + 2\sum_{j=1}^{z} w(j,z)\tilde{\rho}_j \end{aligned}}$$

An alternative approach is to realize that

$$Var\left(\frac{1}{\sqrt{T}}\sum_1^T v_t\right) \quad \text{when } T \to \infty$$

is the LRun Variance of v_t.

How do we calculate Long Run Variance in this course?

- We assume $\hat{v}_t = x_t \hat{u}_t$ follows an AR(p)

$$(1 - \phi_1 L - \dots \phi_p L^p) v_t = \varepsilon_t$$

$$v_t = \phi_p^{-1}(L)\,\varepsilon_t$$

$$\text{with } \phi_p^{-1}(L) = \frac{1}{1 - \phi_1 L - \dots - \phi_p L^p} = C(L)$$

$$v_t = C(L)\,\varepsilon_t$$
$$= C(1)\,\varepsilon_t + (1-L)\,\tilde{C}(L)\,\varepsilon_t$$

- LRV of v_t

$$\frac{1}{\sqrt{T}}\sum v_t = C(1)\frac{\sum_1^T \varepsilon_t}{\sqrt{T}} + o_p(1)$$

$$V\left(\frac{1}{\sqrt{T}}\sum_1 v_t\right) = \boxed{C^2(1)\,\sigma_\varepsilon^2} \quad \text{when } T \to \infty$$

So $\boxed{V(\hat{\beta}_T) = \frac{1}{T}\frac{C^2(1)\,\sigma_\varepsilon^2}{(\sigma_x^2)^2}}$