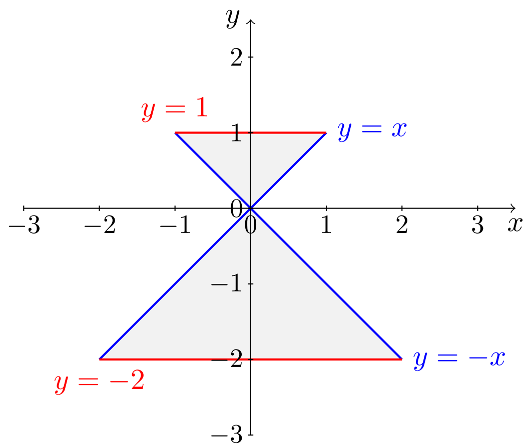
UC3M
Mathematical Optimization for Economics
Final Exam, May 15 2025 **Solutions**

1

On the set $A = \{(x,y) \in \mathbb{R}^2 : |x| \le y \le 1\} \cup \{(x,y) \in \mathbb{R}^2 : -2 \le y \le -|x|\}$ whose representation is given in the figure below the Pareto order is considered.



(a) (10 points) Find the maximal points, minimal points, the maximum and the minimum of A, if they exist. Justify your answers.

(b) (10 points) Consider the function $f(x,y) = x^2 + y^2$ defined on the set A. Using the level curves of the function $f(x,y)$, and the directions of maximum (minimum) growth, identify, giving a graphical reasoning, the local extrema and the global extrema of f on A.

Solution:

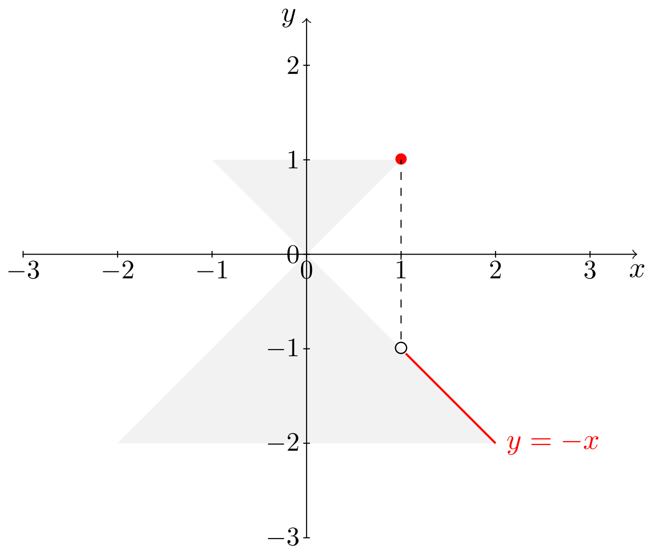
(a) In the figure below, to the left, the maximal points of A are represented

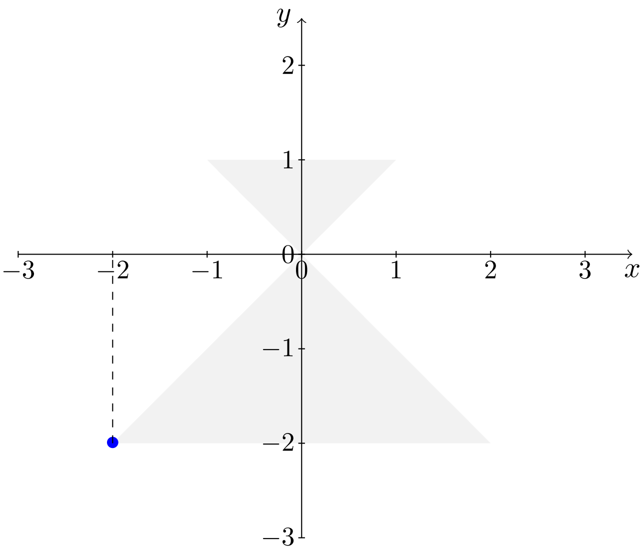
$$\text{maximals}(A) = \{(1,1)\} \cup \{(x,y) \in \mathbb{R}^2 : y = -x, 1 < x \le 2\}.$$

and to the right, the minimum, $(-2,-2)$; therefore, A has no maximum. Obviously,

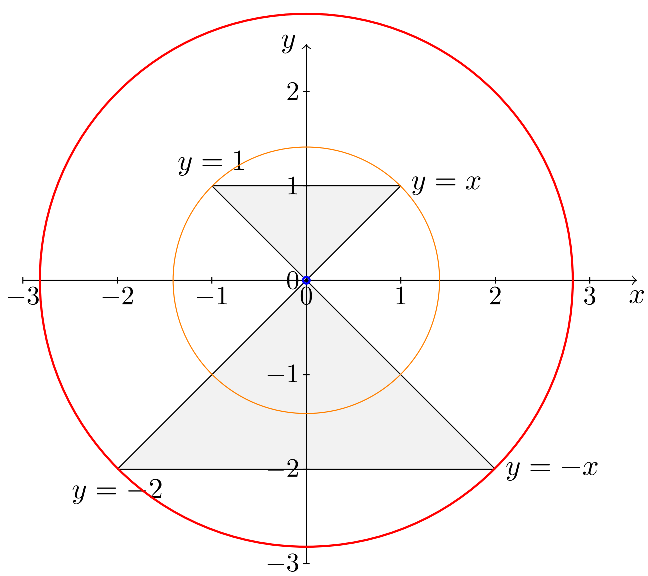
$$\text{minimals}(A) = \{(-2,-2)\}.$$

Note that $(1,-1)$ is not a maximal point of A, since it is dominated in the Pareto sense by $(1,1)$.





(b) The level curves of f are circles $x^2 + y^2 = r^2$ with radius r. Clearly, the value of f increases with the radius (the value is r^2). The largest level curve that intersects A is the circle of radius r that does so at the points $(-2, -2)$ and $(2, -2)$, which are global maxima of f on A. To find r, although not necessary, note that $f(-2, -2) = 4 + 4 = 8$, so $r = \sqrt{8}$. The global minimum is reached at $(0, 0)$, which is the (degenerate) circle of radius 0. The points $(-1, 1)$ and $(1, 1)$ are local maximum.



2

Consider the function $f(x, y) = x^2 + 3y^4 - 4y^3$.

(a) (10 points) Find and classify its local extrema.

(b) (10 points) Does f attain a global minimum on the set $A = \left\{(x, y) \in R^2 \,:\, y > \frac{3}{4}\right\}$? Justify your answer.

Solution:

(a) The necessary conditions $\nabla f = (0, 0)$ provide the equations

$$\begin{cases} 2x = 0 \\ 12y^3 - 12y^2 = 0 \end{cases}$$

their solutions are $(0, 0)$ and $(0, 1)$.

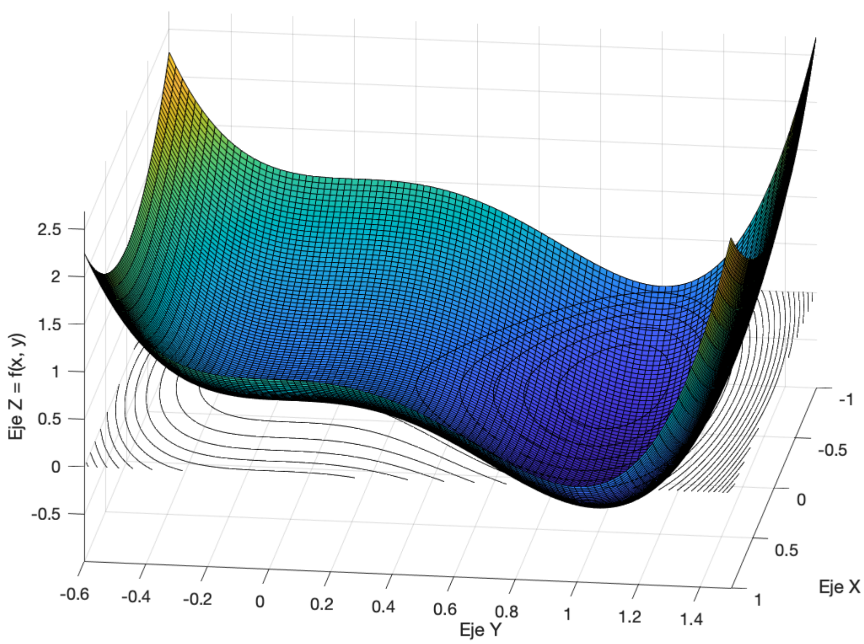
For the two critical points obtained, the second-order sufficient conditions are checked by substituting into the Hessian

$$Hf(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 36y^2 - 24y \end{pmatrix}$$

and we obtain $Hf(0, 1) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$ and $Hf(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

Therefore, at the point $(0, 1)$, f attains a local minimum (positive definite). At the point $(0, 0)$, the Hessian criterion is inconclusive (positive semidefinite). We consider the restriction $f(0, y) = 3y^4 - 4y^3$ to see if it attains an extremum or not. Taking the derivative with respect to y, $f'(0, y) = 12y^3 - 12y^2 = 12y^2(y - 1)$, so according to the first derivative test applied to $f'(0, y)$, the restriction is increasing at $y = 0$, therefore $(0, 0)$ is not an extremum but a saddle point.

The following figure shows the graph of the function around the points $(0, 0)$ and $(0, 1)$.



(b) Since $(0,1) \in A$ and A is an open and unbounded set, we study the convexity of f by checking when the Hessian is positive definite.

$$Hf(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 36y^2 - 24y \end{pmatrix}$$

It is sufficient to check when $2(36y^2 - 24y) > 0$, that is $2y(36y - 24) > 0$, from which $y > \frac{24}{36} = \frac{2}{3}$. Therefore, f is convex on A and $(0,1)$ is a global minimum of f on A.

3

Consider the following Lagrange problem:

$$\text{Optimize } \left\{ f(x,y,z) = 4x^{\frac{1}{2}}yz \right\} \text{ subject to: } 2x + y + 2z = 10,\ x > 0.$$

(a) (10 points) Write the Lagrangian of the problem and find all its critical points.

(b) (15 points) Apply the sufficient conditions to the obtained critical point(s) and classify them.

(c) (10 points) Give an estimation of the optimal value of the objective function when the constraint changes from $2x + y + 2z = 10$ to $2x + y + 2z = 9.5$.

Solution:

(a) $L(x,y,z,\lambda) = 4x^{\frac{1}{2}}yz + \lambda(10 - 2x - y - 2z)$ is the Lagrangian. $\nabla L(x,y,z,\lambda) = \mathbf{0}$ is the necessary condition for optimality, which unfolds to

$$\begin{aligned} L_x &= 2\frac{yz}{x^{\frac{1}{2}}} - 2\lambda = 0 \\ L_y &= 4x^{\frac{1}{2}}z - \lambda = 0 \\ L_z &= 4x^{\frac{1}{2}}y - 2\lambda = 0 \\ L_\lambda &= 10 - 2x - y - 2z = 0 \end{aligned}$$

From the first 3 equations, the following equalities are obtained

$$\boxed{\lambda = \frac{yz}{x^{\frac{1}{2}}} = 4x^{\frac{1}{2}}z = 2x^{\frac{1}{2}}y}$$

- If $y = 0$, then $z = 0$ and, reciprocally, if $z = 0$ then $y = 0$. In this case we obtain the solution

$$(x,y,z) = (5,0,0),\ \lambda = 0.$$

- Let's look for solutions with $y \neq 0$, $z \neq 0$.
We have $yz = 4xz$ from the second equality and dividing by z, $y = 4x$. The third equality is $4x^{\frac{1}{2}}z = 2x^{\frac{1}{2}}y$ and simplifying, $y = 2z$. Taking into account the above, $y = 4x = 2z$, we have $z = 2x$. Now, substituting into the constraint, $10 = 2x + y + 2z = 2x + 4x + 4x = 10x$, we find $x = 1$, $y = 4$, $z = 2$. We have thus obtained the critical point $(1,4,2)$ with a multiplier value of $\lambda = 8$.

(b) Hessian of the Lagrangian:

$$HL(x,y,z,\lambda) = \begin{pmatrix} -\frac{yz}{x^{\frac{3}{2}}} & \frac{2z}{x^{\frac{1}{2}}} & \frac{2y}{x^{\frac{1}{2}}} \\ \frac{2z}{x^{\frac{1}{2}}} & 0 & 4x^{\frac{1}{2}} \\ \frac{2y}{x^{\frac{1}{2}}} & 4x^{\frac{1}{2}} & 0 \end{pmatrix}$$

- At the critical point $(5, 0, 0)$:

$$HL(5,0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{5} \\ 0 & 4\sqrt{5} & 0 \end{pmatrix}$$

  which is indefinite. The quadratic form is

$$Q(x,y,z) = 8\sqrt{5}yz$$

  We must restrict Q to the tangent plane. Since the constraint is a plane, its tangent plane at the origin of coordinates is $2x + y + 2z = 0$. Taking $y = -2(x + z)$ in Q we find the restricted quadratic form

$$q(x,z) = -16\sqrt{5}(x+z)z = -16\sqrt{5}xz - 16\sqrt{5}z^2.$$

  The matrix of q is

$$\begin{pmatrix} 0 & -8\sqrt{5} \\ -8\sqrt{5} & -16\sqrt{5} \end{pmatrix}$$

  which is indefinite, so $(5, 0, 0)$ is a saddle point.

- At the critical point $(1, 4, 2)$:

$$HL(1,4,2,8) = \begin{pmatrix} -8 & 4 & 8 \\ 4 & 0 & 4 \\ 8 & 4 & 0 \end{pmatrix}$$

  which is indefinite. The quadratic form is

$$Q(x,y,z) = -8x^2 + 8xy + 16xz + 8yz$$

  We must restrict Q to the tangent plane. As mentioned before, its tangent plane at the origin of coordinates is $2x + y + 2z = 0$. Taking $y = -2(x + z)$ in Q we find the restricted quadratic form

$$\begin{aligned} q(x,z) &= -8x^2 - 16x(x+z) + 16xz - 16(x+z)z \\ &= -24x^2 - 16xz - 16z^2. \end{aligned}$$

  The matrix of q is

$$\begin{pmatrix} -24 & -8 \\ -8 & -16 \end{pmatrix}$$

  which is negative definite, so $(1, 4, 2)$ is a local maximum.

(c) The value at the optimum is $f(1, 4, 2) = 4 \times 4 \times 2 = 32$ and the value of the multiplier is $\lambda = 8$. Since $\Delta b = -0.5$ (b is the constant term of the constraint), the optimal value decreases by approximately $8 \times 0.5 = 4$ units, so we can estimate that with the new constraint the optimal value of f will be approximately $32 - 4 = 28$.

4

Consider the program

$$\begin{array}{ll} \text{Minimize} & -x-2y \\ \text{s.t.} & x^2+2y^2 \leq 3. \end{array}$$

(a) (15 points) Obtain the solutions of the corresponding Kuhn-Tucker equations for the program.

(b) (10 points) Justify that the program has a global solution. (Note: This part can be answered without doing the first one).

Solution:

(a) Writing the problem in standard form

$$\begin{array}{ll} \text{Maximize} & x+2y \\ \text{s.t.} & x^2+2y^2 \leq 3 \end{array}$$

The Lagrangian is $L(x, y, \lambda) = x + 2y + \lambda(3 - x^2 - 2y^2)$.

The K-T necessary conditions are the following equations and inequalities:

$$1 - 2\lambda x = 0 \tag{1}$$
$$2 - 4\lambda y = 0 \tag{2}$$
$$\lambda(3 - x^2 - 2y^2) = 0 \tag{3}$$
$$x^2 + 2y^2 \leq 3 \tag{4}$$
$$\lambda \geq 0 \tag{5}$$

Considering the first three equations we have that:

$$\begin{aligned} 1 - 2\lambda x &= 0 \\ 2 - 4\lambda y &= 0 \\ \lambda(3 - x^2 - 2y^2) &= 0 \end{aligned}$$

- If in equation (3) we set $\lambda = 0$, we get $1 = 0$.
- If in equation (3) we set $\lambda > 0$, this forces $x^2 + 2y^2 = 3$ and we get

$$\begin{aligned} 1 - 2\lambda x &= 0 \\ 2 - 4\lambda y &= 0 \\ x^2 + 2y^2 &= 3 \end{aligned}$$

Subtracting the second equation from the first equation multiplied by 2, we get $-4\lambda(y - x) = 0$, from which $y = x$, and substituting into the third equation we have $x^2 + 2x^2 = 3$ with solution $x = \pm 1$. With $x = 1$ and $y = 1$, $\lambda = \frac{1}{2}$, $(1, 1, \frac{1}{2})$ which is a solution of the K-T equations. The solution $x = -1$ and $y = -1$ gives $\lambda = -\frac{1}{2}$ which does not satisfy (5)

Therefore, the unique solution of the K-T system of equations is $(1, 1, \frac{1}{2})$

(b) The function is continuous as it is a polynomial and the feasible region is compact as it is closed (with inequalities $\leq$) and bounded: $x^2 \leq x^2 + 2y^2 \leq 3$ and $y^2 \leq x^2 + 2y^2 \leq 3$ so $x^2 + y^2 \leq 6$. We can apply Weierstrass' theorem to affirm that the critical point is a global minimum of the original program.

We can also justify it because the original program is convex and the critical point becomes a global solution. Indeed, the objective function is linear and therefore convex (and concave), the feasible region is $C = \left\{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 \leq 3\right\} = \left\{(x, y) \in \mathbb{R}^2 : h(x, y) \leq 3\right\}$ with $h(x, y) = x^2 + 2y^2$ is also convex because h is convex in $\mathbb{R}^2$ since

$$Hh(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

is positive definite.