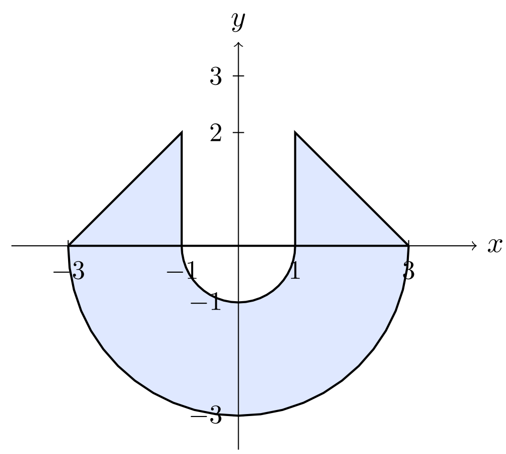
UC3M
Mathematical Optimization for Economics
Extraordinary Final Exam July 1, 2025 Solutions

1

Consider the set A defined as

$$\left\{(x,y)\in\mathbb{R}^2 : y\leq 0,\ 1\leq x^2+y^2\leq 9\right\}\cup\left\{(x,y)\in\mathbb{R}^2 : y\geq 0,\ 1\leq |x|\leq 3,\ y\leq 3-|x|\right\}$$

ordered with the Pareto order $\leq_P$ (Recall that $(x,y)\leq_P(a,b)\Leftrightarrow x\leq a$ and $y\leq b$). The set is represented in the following figure:

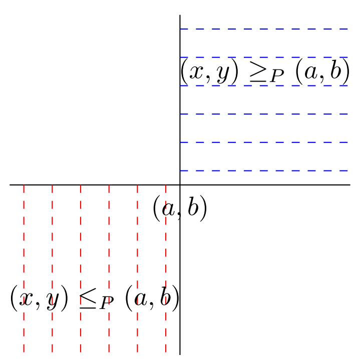


(a) (10 points) Find the maximal, minimal points, the maximum and the minimum of A, if they exist. Justify your answers.

(b) (10 points) On set A, consider the function defined by $f(x,y)=y-x$. Justify that f has a maximum and a minimum. Then, using a graphical reasoning on the level curves, find the points at which the maximum and minimum of f are reached and calculate their values.

Solution:

(a) The maximal and minimal points must be on the boundary of A. Graphically, using the cross strategy, i.e., superimposing preference cones on points (a,b) on the boundary of A as shown in this figure:



it can be concluded that

$$\text{maximal}(A)=\{(x,y)\in A : 1\leq x\leq 3,\ x+y=3\}$$

$$\text{minimal}(A) = \{(x, y) \in A : x^2 + y^2 = 9,\ y \leq 0,\ x \leq 0\}$$

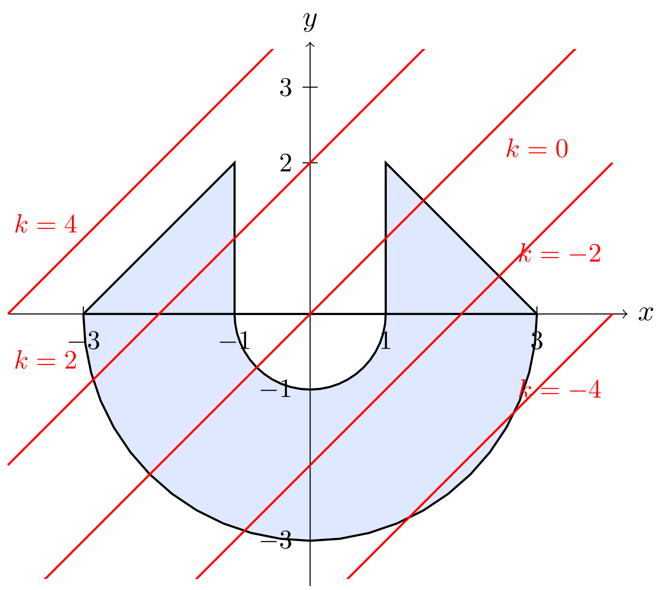
Since the set of maximal points of A is not reduced to a single point, we can deduce that A has no maximum. For the same reason, that is, because the set of minimal points of A is not reduced to a single point, A has no minimum.

(Note: that A has a single maximal (minimal) point does not imply the existence of a maximum (minimum)).

(b) The set A is a compact set and f is continuous. Therefore, f attains its maximum and minimum.

The function f is increasing with respect to the second coordinate and decreasing with respect to the first coordinate. Therefore f attains its maximum and minimum on ∂A. The level curve k is

$$\mathcal{C}_k(f) = \{(x, y) \in A : y - x = k\}$$



That is, the level curves are parallel lines. We know that for a point p to be a maximum or minimum, the level curve passing through p cannot penetrate the interior of A. Therefore, either the level curve is tangent to the boundary of A, or point p is a corner of A.

According to the drawing, the maximum of f is reached at the points of the set

$$\{(x, y) \in A : -3 \leq x \leq -1,\ y - x = 3\}$$

and the maximum of f is 3. On the contrary, the minimum is reached at a tangency point in the fourth quadrant. By symmetry, this point must be $\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$ and the minimum of f is $-3\sqrt{2}$.

2

Consider the function $f(x,y) = -y^2 - 3x^4 + 4x^3$.

(a) (10 points) Find the largest open convex set in $\mathbb{R}^2$ where f is strictly concave.

(b) (10 points) Find and classify the critical points of f.

Solution:

(a) The Hessian of f is

$$Hf(x,y) = \begin{pmatrix} -36x^2 + 24x & 0 \\ 0 & -2 \end{pmatrix}$$

The principal minors are $-36x^2 + 24x = 12x(2-3x)$ and $-24x(2-3x)$. Notice that they have alternate signs starting with negative if and only if $x(2-3x) < 0$. This product is negative if and only if $x < 0$ or $x > 2/3$. Thus, consider the sets

$$C_1 = \{(x,y) \in \mathbb{R}^2 : x < 0\} \quad \text{and} \quad C_2 = \{(x,y) \in \mathbb{R}^2 : x > 2/3\}.$$

Both are open and convex, and the function $f(x,y)$ is strictly concave in each of them. However, the union $C_1 \bigcup C_2$ is not a convex set, thus f is not strictly concave on the union. Thus, there is no a largest open and convex set where f is strictly concave.

(b) The necessary conditions $\nabla f = (0,0)$ provide the system

$$\begin{cases} -12x^3 + 12x^2 = 0 \\ -2y = 0 \end{cases}$$

whose solutions are $(0,0)$ and $(1,0)$. Since $(1,0) \in C_2$, where C_2 was defined in the previous section, and where f is strictly concave, we see that $(1,0)$ is a global maximum (unique) in the open set C_2. Therefore, it is a local maximum in $\mathbb{R}^2$. Obviously, this conclusion can also be reached by studying the sign of $Hf(1,0)$. As for $(0,0)$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

is negative semidefinite. The sufficient conditions are not applicable. However, $f(x,0) = -3x^4 + 4x^3 = x^3(-3x+4)$ takes positive and negative values in any interval of x centered at 0. Therefore, $(0,0)$ is a saddle point of f.

3

Consider the following Lagrange problem:

Optimize $f(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 2xy - 7x - 9y - 27z$
subject to: $x + y + z = 10$, with $x, y, z \in \mathbb{R}$.

(a) (15 points) Write the Lagrangian of the problem and find all its critical points.

(b) (10 points) Apply the sufficient conditions to the critical points obtained in the previous section and classify them using the Hessian matrix of the Lagrangian. Can we affirm that any solution is global? Justify your answer.

(c) (5 points) Approximately how much does the optimal value of the objective function change when the constraint changes from $x + y + z = 10$ to $x + y + z = 9.5$?

Solution:

The objective function is $f(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 2xy - 7x - 9y - 27z$.

The constraint is $g(x,y,z) = x + y + z - 10 = 0$.

(a) The Lagrangian function $L(x,y,z,\lambda)$ is:

$$L(x,y,z,\lambda) = 2x^2 + 2y^2 + 2z^2 + 2xy - 7x - 9y - 27z - \lambda(x + y + z - 10)$$

To find the critical points, we set the partial derivatives of L with respect to x, y, z, and λ to zero:

$$\frac{\partial L}{\partial x} = 4x + 2y - 7 - \lambda = 0 \quad \Rightarrow \quad \lambda = 4x + 2y - 7 \tag{1}$$
$$\frac{\partial L}{\partial y} = 4y + 2x - 9 - \lambda = 0 \quad \Rightarrow \quad \lambda = 4y + 2x - 9 \tag{2}$$
$$\frac{\partial L}{\partial z} = 4z - 27 - \lambda = 0 \quad \Rightarrow \quad \lambda = 4z - 27 \tag{3}$$
$$\frac{\partial L}{\partial \lambda} = -(x + y + z - 10) = 0 \quad \Rightarrow \quad x + y + z = 10 \tag{4}$$

Equating (1) and (2): $4x + 2y - 7 = 4y + 2x - 9$; $2x - 2y = -2$; $x - y = -1$
$\Rightarrow \quad y = x + 1$ (Equation A)

Equating (2) and (3): $4y + 2x - 9 = 4z - 27$; $2x + 4y - 4z = -18$;
$\Rightarrow \quad x + 2y - 2z = -9$ (Equation B)

Substitute Equation A into Equation B: $x + 2(x+1) - 2z = -9$; $x + 2x + 2 - 2z = -9$; $3x + 2 - 2z = -9$; $3x - 2z = -11$ (Equation C)

From constraint (4) and Equation A, express z in terms of x: $x + (x+1) + z = 10$ $2x + 1 + z = 10$ $\Rightarrow \quad z = 9 - 2x$ (Equation D)

Substitute Equation D into Equation C: $3x - 2(9 - 2x) = -11$; $3x - 18 + 4x = -11$; $7x = 7$; $x^* = 1$

Now find y^* and z^* using $x^* = 1$: $y^* = x^* + 1 = 1 + 1 = 2$ $z^* = 9 - 2x^* = 9 - 2(1) = 7$

The only critical point is $(x^*, y^*, z^*) = (1, 2, 7)$.

The value of λ^* at this critical point is: $\lambda^* = 4x^* + 2y^* - 7 = 4(1) + 2(2) - 7 = 4 + 4 - 7 = 1$.

(b) The Hessian matrix of the Lagrangian with respect to x, y, z is H_L:
$\frac{\partial^2 L}{\partial x^2} = 4 \quad \frac{\partial^2 L}{\partial x \partial y} = 2$
$\frac{\partial^2 L}{\partial y^2} = 4 \quad \frac{\partial^2 L}{\partial x \partial z} = 0$
$\frac{\partial^2 L}{\partial z^2} = 4 \quad \frac{\partial^2 L}{\partial y \partial z} = 0$
Thus, the Hessian matrix of the Lagrangian is:

$$H_L(x, y, z) = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

To verify convexity (positive definiteness of H_L), we examine its leading principal minors:

- $\Delta_1 = \det(4) = 4 > 0$
- $\Delta_2 = \det \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = (4)(4) - (2)(2) = 16 - 4 = 12 > 0$
- $\Delta_3 = \det \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4 \cdot \det \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = 4 \cdot 12 = 48 > 0$

Since all leading principal minors are positive, the Hessian matrix H_L is positive definite. This implies that the objective function $f(x, y, z)$ is **strictly convex**.

For an optimization problem with a strictly convex objective function and a linear constraint, any critical point obtained by the Lagrangian method (KKT conditions) is guaranteed to be the **unique global minimum**. Therefore, the critical point $(1, 2, 7)$ is the **global minimum**.

(c) The Lagrange multiplier λ^* provides an approximation of the change in the optimal value of the objective function (Δf^*) for a small change in the constraint constant (Δb). The relationship is:

$$\Delta f^* \approx \lambda^* \cdot \Delta b$$

- Change in the constraint constant (Δb): $b_2 - b_1 = 9.5 - 10 = -0.5$
- Optimal Lagrange multiplier (λ^*): 1 (from part a)

The change in the optimal value is approximately:

$$\Delta f^* \approx \lambda^* \cdot \Delta b = 1 \cdot (-0.5) = -0.5.$$

4

Consider the program

$$\begin{array}{ll} \text{Maximize} & f(x,y) = -x^2 - y^2 + 6y \\ \text{subject to:} & (x,y) \in A, \end{array}$$

where the feasible set A is given by two inequalities:

$$A = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 \leq y,\; y \leq 2x+3\}.$$

(a) (15 points) Obtain the solutions of the Kuhn–Tucker equations corresponding to the program.

(b) (5 points) Are the points found in part (a) local and/or global maxima? Justify your answer.

(c) (10 points) What happens if we substitute the feasible set A with the following feasible set B:

$$B = \{(x,y) \,:\, x \geqslant -1,\; y \geqslant 2\}?$$

Is it necessary to solve the problem again to obtain the solution? Justify your answer.

Observation: it is recommended to draw sets A and B.

Solution:

(a) The Lagrangian of the problem is

$$L(x,y,\lambda,\mu) = -x^2 - y^2 + 6y + \lambda(y - x^2) + \mu(2x - y + 3).$$

The Kuhn–Tucker equalities are:

$$\frac{\partial L}{\partial x}(x,y,\lambda,\mu) = -2x - 2\lambda x + 2\mu = 0 \tag{1}$$

$$\frac{\partial L}{\partial y}(x,y,\lambda,\mu) = -2y + 6 + \lambda - \mu = 0 \tag{2}$$

$$\lambda(y - x^2) = 0 \tag{3}$$

$$\mu(2x - y + 3) = 0 \tag{4}$$

The KT inequalities are:

$$\lambda \geqslant 0 \tag{5}$$

$$\mu \geqslant 0 \tag{6}$$

$$x^2 \leq y \tag{7}$$

$$y \leq 2x + 3 \tag{8}$$

Next, we will find the solutions to the system formed by equations (1)–(4), organizing the search considering the multiplicative structure of the system. For these solutions, we will check if inequalities (5)–(8) are satisfied.

Consider the following cases:

(i) $\lambda, \mu > 0 \Longrightarrow (x, y) = (-1, 1)$ or $(x, y) = (3, 9)$ by equations (3) and (4). However:

- $\frac{\partial L}{\partial x}(-1, 1, \lambda, \mu) = 0 \Longleftrightarrow 2\lambda + 2\mu = -2$, in contradiction with (5) and (6).
- $\begin{cases} \frac{\partial L}{\partial x}(3, 9, \lambda, \mu) = 0 \Longleftrightarrow -6\lambda + 2\mu = 6 \\ \frac{\partial L}{\partial y}(3, 9, \lambda, \mu) = 0 \Longleftrightarrow \lambda - \mu = 12. \end{cases}$

  Multiplying the second equation by 2 and summing to the first one, we get $-4\lambda = 30$, in contradiction with (5).

(ii) $\lambda, \mu = 0 \Longrightarrow (x, y) = (0, 3)$ satisfies all conditions (1)–(8).

(iii) $\lambda > 0, \mu = 0 \Longrightarrow$
$\frac{\partial L}{\partial x}(x, y, \lambda, \mu) = -2x - 2\lambda x = 0 \Longrightarrow x = 0$, because, otherwise, $\lambda = -1$ contradicts (5).
But $x = 0$ implies, by (3), $y = 0$; then, by (2) $\lambda = -6$ contradicts (5).

(iv) $\lambda = 0, \mu > 0 \Longrightarrow$

$$\frac{\partial L}{\partial x}(x, y, \lambda, \mu) = -2x + 2\mu = 0 \Longrightarrow x = \mu$$
$$\frac{\partial L}{\partial y}(x, y, \lambda, \mu) = -2y + 6 - \mu = 0 \Longrightarrow y = 3 - \mu/2$$
$$\mu(2x - y + 3) = 0 \Longrightarrow 2\mu - 3 + \mu/2 + 3 = 0 \Longrightarrow \mu = 0,$$

contradicting (iv).

Therefore, $(0, 3, 0, 0)$ is the only point that satisfies the Kuhn–Tucker conditions.

(b) Set A is compact, the function is continuous, and there is only one candidate for a local maximum. Therefore, by Weierstrass's theorem, the point $(0, 3)$ will be a local and global maximum in A.

Another way: The set is convex, the function is strictly concave, thus the critical point $(0, 3)$ is a global maximum in A.

(c) Set B is convex and the function is strictly concave. The critical point for the function $f(x, y)$ without constraints is $(0, 3)$, which belongs to set B, thus it is a global maximum in B.