1

Consider the function $f(x,y) = \dfrac{1}{y - \frac{x}{2} + 1}$ and the set

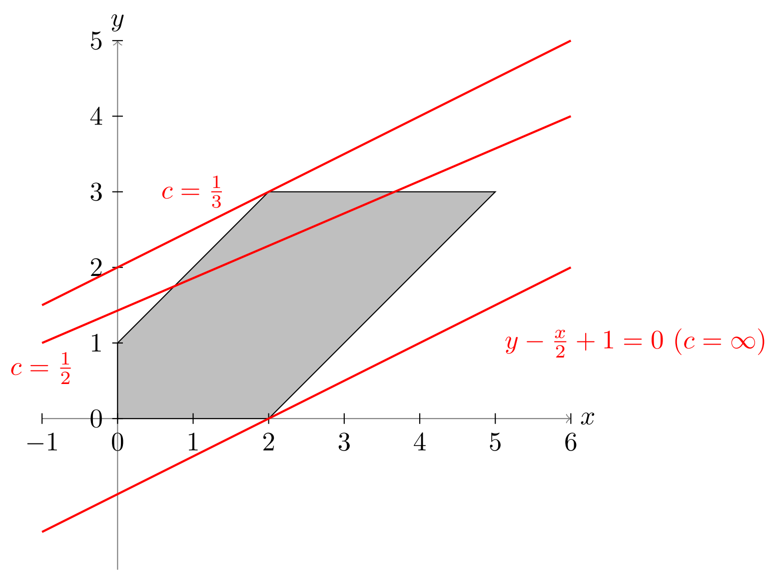
$$A = \{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0,\ y \geq 0,\ y \leq 3,\ x - 2 \leq y \leq x + 1\}.$$

(a) (6 points) Draw the set A and discuss whether the function f and the set A satisfy the assumptions of the Theorem of Weierstrass. Can you ensure the existence of global extremes of f in A?

(b) (6 points) Draw some of the level curves of f on the plane, showing the directions in which f increases/decreases and determine (if they exist) the global extrema of f on A. In case they do not exist, justify your answer.

Solution:

(a) Below is a representation of the set A, join with 2 level curves, of levels $\frac{1}{2}$ and $\frac{1}{3}$, respectively. Note that the function is not defined on the line $y - \frac{x}{2} + 1 = 0$.



The set A is closed (contain its boundary) and bounded, thus compact. The function f is not defined along the line $y - \frac{x}{2} + 1 = 0$; this line touches the set A at the point $(2,0)$. Thus, f is not continuous in A and we cannot apply the Theorem of Weierstrass. In consequence, the existence of global extrema cannot be assured.

(b) The level curves of f are given by

$$\frac{1}{y - \frac{x}{2} + 1} = c, \quad c \in \mathbb{R} \Rightarrow y - \frac{x}{2} + 1 = \frac{1}{c} \neq 0,$$

They are lines of the same slope $\frac{1}{2}$, thus they are parallel lines with different ordinate, which grows as c (the level of f) diminishes. Note that $y - \frac{x}{2} + 1 > 0$ for any $(x,y) \in A$, hence $c > 0$. This means that $f(x,y)$ grows to $+\infty$ as $y - \frac{x}{2} + 1$ approaches 0 from points in the set A. Thus

$$\lim_{\substack{(x,y)\to(2,0)\\(x,y)\in A}} f(x,y) = \lim_{\substack{(x,y)\to(2,0)\\(x,y)\in A}} \frac{1}{y - \frac{x}{2} + 1} = +\infty,$$

hence f is not bounded above on A and has no global maximum. On the other hand, the line $y - \frac{x}{2} + 1 = \frac{1}{c}$ of highest ordinate at the origin, $\frac{1}{c} - 1$, and hence, of minimum level c, touches the set A at $(2,3)$. Thus, $(2,3)$ is the global minimum of f on A, and the minimum value is $f(2,3) = \frac{1}{3}$.

2

Consider the function $f(x, y) = (x^2 - x + 1)e^{x+ay^2}$, where a is an unknown parameter, different from 0.

(a) (6 points) Find the critical points of f.

(b) (6 points) For each of the critical points found in the item above, determine the range of values of the parameter a for which the critical point considered is

- A local maximum.
- A local minimum.
- A saddle point.

(c) (6 points) Prove that the function f has no global maximum or global minimum.
Hint: Consider the function $g(x) = f(x, 0) = (x^2 - x + 1)e^x$ *and calculate* $\lim_{x\to\pm\infty}(x^2 - x + 1)e^x$.

Solution:

(a) Critical points are those where either the function is not differentiable or the gradient is the null vector.

$$\nabla f(x, y) = \left(\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y}\right) = ((x^2 + x)e^{x+ay^2}, 2ay(x^2 - x + 1)e^{x+ay^2})$$

$$\nabla f(x, y) = 0 \Rightarrow \left\{ \begin{array}{c} (x^2 + x)e^{x+ay^2} = 0 \\ 2ay(x^2 - x + 1)e^{x+ay^2} = 0 \end{array} \right\}$$

or, equivalently $x^2 + x = 0$ and $y = 0$. This is because both $e^{x+ay^2} \neq 0$ and $x^2 - x + 1 \neq 0$. Thus, we obtain two critical points, $(0, 0)$ and $(-1, 0)$.

(b) To classify the critical points, we use the second order sufficient conditions, which depend on the sign of the quadratic form associated to the Hessian matrix of f, $\mathrm{H}f(x, y)$:

$$\frac{\partial^2 f(x, y)}{\partial x^2} = (x^2 + 3x + 1)e^{x+ay^2}$$

$$\frac{\partial^2 f(x, y)}{\partial y \partial x} = 2ay(x^2 + x)e^{x+ay^2}$$

$$\frac{\partial^2 f(x, y)}{\partial y^2} = (x^2 - x + 1)(4a^2y^2 + 2a)e^{x+ay^2}$$

$$\mathrm{H}f(x, y) = e^{x+ay^2} \begin{pmatrix} x^2 + 3x + 1 & 2ay(x^2 + x) \\ 2ay(x^2 + x) & (x^2 - x + 1)(4a^2y^2 + 2a) \end{pmatrix}$$

At $(0, 0)$ we get

$$\mathrm{H}f(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 2a \end{pmatrix},$$

a diagonal matrix.

- $\mathrm{H}f(0,0)$ es positive definite iff $a > 0$ thus $(0,0)$ is a local minimum in this case.
- $\mathrm{H}f(0,0)$ is indefinite iff $a < 0$, thus $(0,0)$ is a saddle point in this case.

At $(-1,0)$ we get

$$\mathrm{H}f(-1,0) = e^{-1}\begin{pmatrix} -1 & 0 \\ 0 & 6a \end{pmatrix},$$

a diagonal matrix.

- $\mathrm{H}f(-1,0)$ is negative definite iff $a < 0$, thus $(-1,0)$ is a local maximum in this case.
- $\mathrm{H}f(-1,0)$ is indefinite iff $a > 0$, thus $(-1,0)$ is a saddle point in this case.

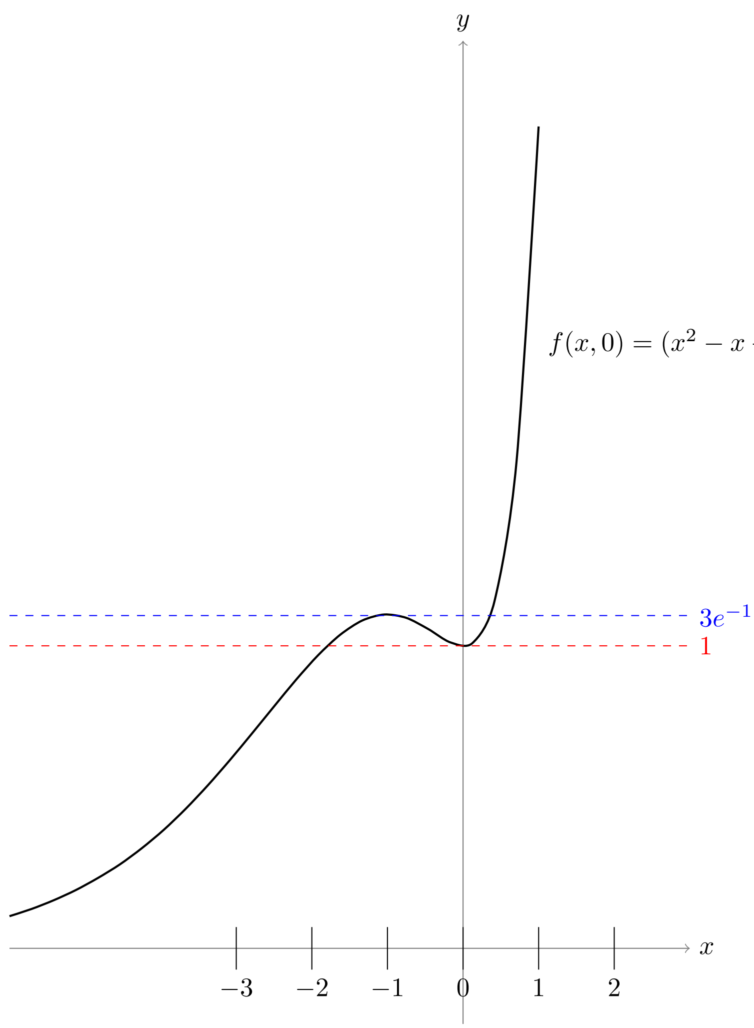
(c) It suffices to observe that $f(x,0)$ grows unboundedly when $x \to \infty$

$$\lim_{x\to\infty}(x^2 - x + 1)e^x = \infty$$

and that

$$\lim_{x\to-\infty}(x^2 - x + 1)e^x = 0.$$

Since that $f(0,0) = 1$ and $f(-1,0) = e^{-1}$, it is clear that none of the critical points can be global minimum, independently of the value of a. See the figure below, where the vertical axis has been scaled.



3

Consider the problem of Lagrange:

$$\text{Opt } f(x,y,z) = -x\ln x - y\ln y - z\ln z \quad \text{s.t.: } x+y+z=1$$

(a) (3 points) Obtain the Lagrange equations.

(b) (3 points) Find the critical points of the Lagrangian.

(c) (3 points) Classify the critical points found in the item above. Are they global extrema of f subject to the constraint? Justify your answer.

(d) (6 points) Suppose that the constraint changes to $x+y+z=0.85$, that is, the problem becomes

$$\text{Opt } f(x,y,z) = -x\ln x - y\ln y - z\ln z \quad \text{s.t.: } x+y+z=0.85.$$

Without solving the problem again, calculate approximately the optimal value of $f(x,y,z)$ after this change. This implies to calculate approximately the value of the multiplier, knowing that $\ln 3 \approx 1.1$.

Solution:

(a) The Lagrangian is $L(x,y,z) = -x\ln x - y\ln y - z\ln z + \lambda(1-x-y-z)$ and the Lagrange equations are

(i) $-\ln x - 1 - \lambda = 0,$

(ii) $-\ln y - 1 - \lambda = 0,$

(iii) $-\ln z - 1 - \lambda = 0.$

(b) Let us solve the Lagrange equations to find the critical points. Eqns. (i)–(iii) above give $x=y=z$, and plugging into the constraint we find $x=y=z=\frac{1}{3}$, with $\lambda = -\ln\frac{1}{3} - 1 = \ln 3 - 1 \approx 0.1$.

(c) The feasible set is not compact but f is strictly concave in its domain. Since the constrained set is convex (a plane), the critical point is a global maximum (and it is unique). To see that f is strictly concave, note that

$$Hf(x,y,z) = \begin{pmatrix} -\frac{1}{x} & 0 & 0 \\ 0 & -\frac{1}{y} & 0 \\ 0 & 0 & -\frac{1}{z} \end{pmatrix}$$

is negative definite in $\{(x,y,z) : x>0, y>0, z>0, x+y+z=1\}$, which is the intersection set of the domain of f with the feasible set.

(d) The global maximum is obtained at $(1/3,1/3,1/3)$, where the value of f is $3 \times \left(-\frac{1}{3}\ln\frac{1}{3}\right) = -\ln\frac{1}{3} = \ln 3$. The multiplier was found to be $\lambda = \ln 3 - 1 \approx 0.1$, thus

$$\Delta \text{ optimal value } \approx 0.1(0.85-1) = -0.015 < 0.$$

In consequence, the new optimal value is approximately $\ln 3 - 0.015 \approx 1.1 - 0.015 = 1.085$.

4

Consider the function $f(x, y) = (x-3)^2 + (y-4)^2$ defined on the set

$$A = \{(x, y) \,:\, x^2 + y^2 \le 1\}.$$

(a) (6 points) Establish the Kuhn–Tucker necessary optimality conditions to the problem

$$\max\ (x-3)^2 + (y-4)^2 \quad \text{subject to } x^2 + y^2 \le 1.$$

(b) (6 points) Find all the solutions of the Kuhn–Tucker conditions established in part (a).

(c) (3 points) Find the global maximum of f on A.

Solution:

(a) The Lagrangian is $L(x, y, \lambda) = (x-3)^2 + (y-4)^2 + \lambda(1 - x^2 - y^2)$. The K–T necessary conditions of optimality are

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2(x-3) - 2\lambda x = 0,\\
\frac{\partial L}{\partial y} &= 2(y-4) - 2\lambda y = 0,\\
\lambda\frac{\partial L}{\partial \lambda} &= \lambda(1 - x^2 - y^2) = 0,\\
&\lambda \ge 0,\\
&1 - x^2 - y^2 \ge 0.
\end{aligned}$$

(b) Consider the system of equalities

(i) $2(x-3) - 2\lambda x = 0$

(ii) $2(y-4) - 2\lambda y$

(iii) $\lambda(1 - x^2 - y^2) = 0$.

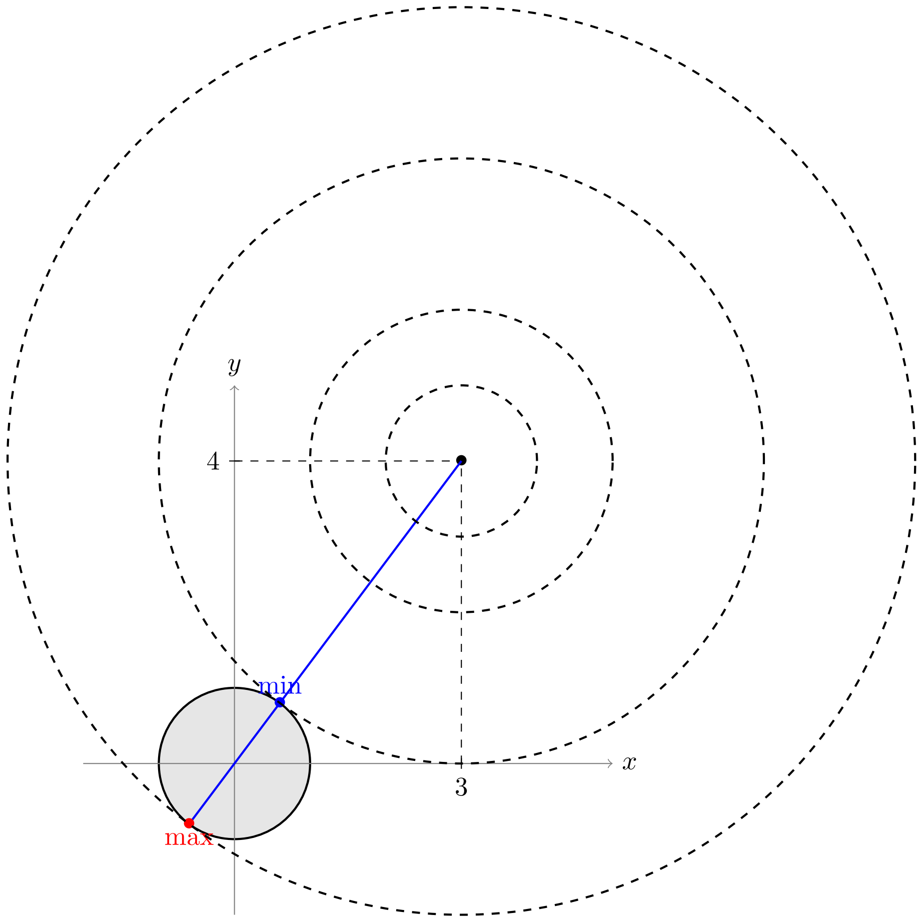
If $\lambda = 0$, then from (i) and (ii) $(x, y) = (3, 4)$. But $3^2 + 4^2 = 25 > 1$, hence $(3, 4)$ is not feasible.

If $\lambda > 0$, then from (i) and (ii) $\lambda = \frac{x-3}{x} = \frac{y-4}{y}$. From this, $1 - \frac{3}{x} = 1 - \frac{4}{y}$, thus $y = \frac{4}{3}x$. Going to (iii), $x^2 + \left(\frac{4}{3}x\right)^2 = 1$, hence $x = \pm\frac{3}{5}$. Thus, $(x, y) = (\frac{3}{5}, \frac{4}{5})$ with $\lambda = 1 - 3/\frac{3}{5} = 1 - 5 = -4 < 0$, and $(x, y) = (-\frac{3}{5}, -\frac{4}{5})$ with $\lambda = 1 + 3/\frac{3}{5} = 1 + 5 = 6 > 0$. Thus, the only point that satisfies KKT is $(x, y) = (-\frac{3}{5}, -\frac{4}{5})$.

Another way to solve the Lagrange system is to realize that $\lambda \ne 1$ and to get $x = \frac{3}{1-\lambda}$ from (i) and $y = \frac{4}{1-\lambda}$ from (ii) and to plug this into the constraint, $\frac{9}{(1-\lambda)^2} + \frac{16}{(1-\lambda)^2} = 1$, hence $(1-\lambda)^2 = 25$ and then $\lambda = 6 > 0$ or $\lambda = -4 < 0$. Choosing the positive solution, we get the critical point $(-\frac{3}{5}, -\frac{4}{5})$.

(c) The function is continuous and the feasible set is compact, thus by the Theorem of Weierstrass, a global maximum exists. Since the maximum must satisfy the K–T necessary conditions, the point determined in the item above is the global maximum.

The figure below is shown for the purpose of illustrating the geometry of the problem. The problem demands to compute the point of the circle $x^2 + y^2 \le 1$ for which the distance to the point $(3, 4)$ is maximum. The picture shows several level curves of function f, which are circumferences centred at $(3, 4)$.



The line joining $(3, 4)$ with $(0, 0)$ cuts A at the two boundary points $(-3/5, -4/5)$ and $(3/5, 4/5)$, which are the global maximum and the global minimum, respectively.