1. There is only one perishable commodity available for consumption today and tomorrow, and two consumers, Ann and Bob, whose preferences for alternative consumption streams today and tomorrow, $(x, y) \in \mathbb{R}^2_+$, are described by the utility functions $u^A(x, y) = \min\{x, y\}$, and $u^B(x, y) = xy$, respectively.

(a) Assume that the endowment streams are (0,30) for Ann and (20,0) for Bob. Ann and Bob participate in a competitive credit market in which they are the only participants. Calculate the competitive equilibrium rate of interest and consumption streams, and determine whether this allocation is Pareto optimal.

Let us normalize (spot) prices: $\hat{P}_x = \hat{P}_y = 1$.

Ann's demand of consumption today:

$$\left.\begin{array}{l} \left.\begin{array}{l} x = b \\ y = 30 - (1+r)b \end{array}\right\} \text{B.C.} \\ \left.\begin{array}{l} x = y \end{array}\right\} \text{OPTIMALITY} \end{array}\right\} \Rightarrow \boxed{b^A(r) = \frac{30}{2+r}}$$

CONSOLIDATING THE B.C: $(1+r)x + y = 30 \iff x + \frac{y}{1+r} = \frac{30}{1+r}$

Bob's demand of consumption today:

$$x = 20 + b$$
$$y = -(1+r)\, b$$
$$\left.\right\} \text{B.C.} \iff (1+r)\,x + y = 20\,(1+r)$$

$$\frac{y}{x} = (1+r) \quad \} \text{ OPTIMALITY}$$

$$\boxed{b^B(r) = -10}$$

ALTERNATIVELY:

$$\boxed{\max_{b \in \mathbb{R}} \; (20+b)\left(-(1+r)\,b\right) \qquad \underline{\text{FOC}}: \; 20 + 2b = 0}$$

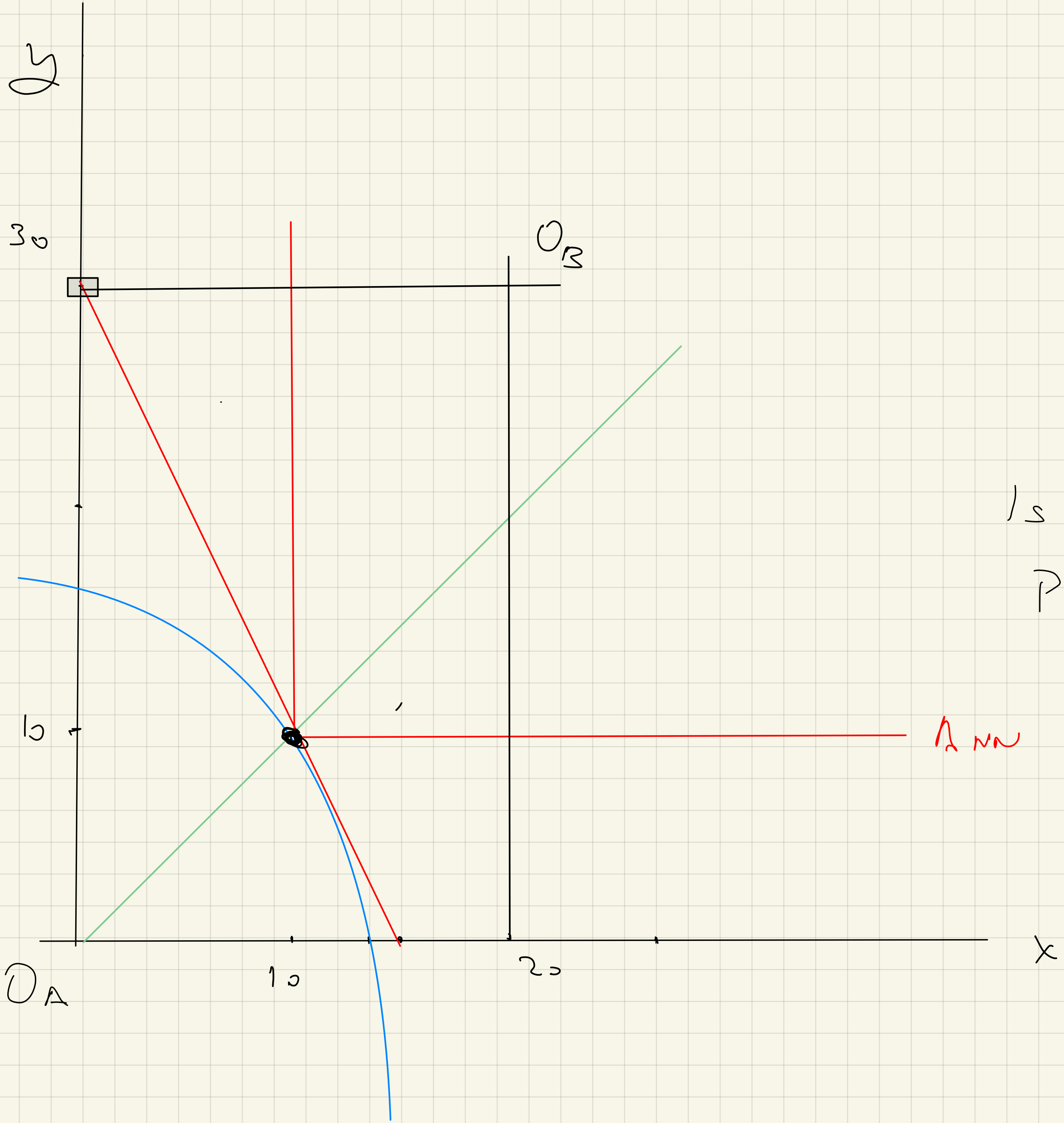
<u>CREDIT MARKET</u> CLEARING

$$b^A(r) + b^B(r) = 0$$

$$\frac{30}{2+r} - 10 = 0 \iff \underline{\underline{r^* = 1}}$$

<u>RADNER</u> CE:

$$(x_A^*, y_A^*) = (10, 10)$$
$$(x_B^*, y_B^*) = (10, 20)$$



IS THE RADNER CE PARETO OPTIMAL?

(b) Now assume that in addition to the credit market there is real investment process which yields three units of the good tomorrow for each unit of good today used as input. The investment process is operated by a competitive firm owned by Ann and Bob in proportions λ and $(1-\lambda)$, where $\lambda \in (0,1)$. Determine the competitive equilibrium interest rates, production plans, and consumption streams.

Firm's Demand of Credit:

$$\max_{b \in \mathbb{R}_+} 3b - (1+r)b \quad \Rightarrow \quad b^F(r) = \begin{cases} 0 & \text{if } 3 < 1+r \\ [0,\infty) & \text{if } 3 = 1+r \\ \infty \text{ (UNDEFINED)} & \text{if } 3 > 1+r. \end{cases}$$

Hence, in a CE: $\quad 3 \leq 1+r^* \iff \left\{ r^* \geq 2 \text{ and } \pi^* = 0 \right\}.$

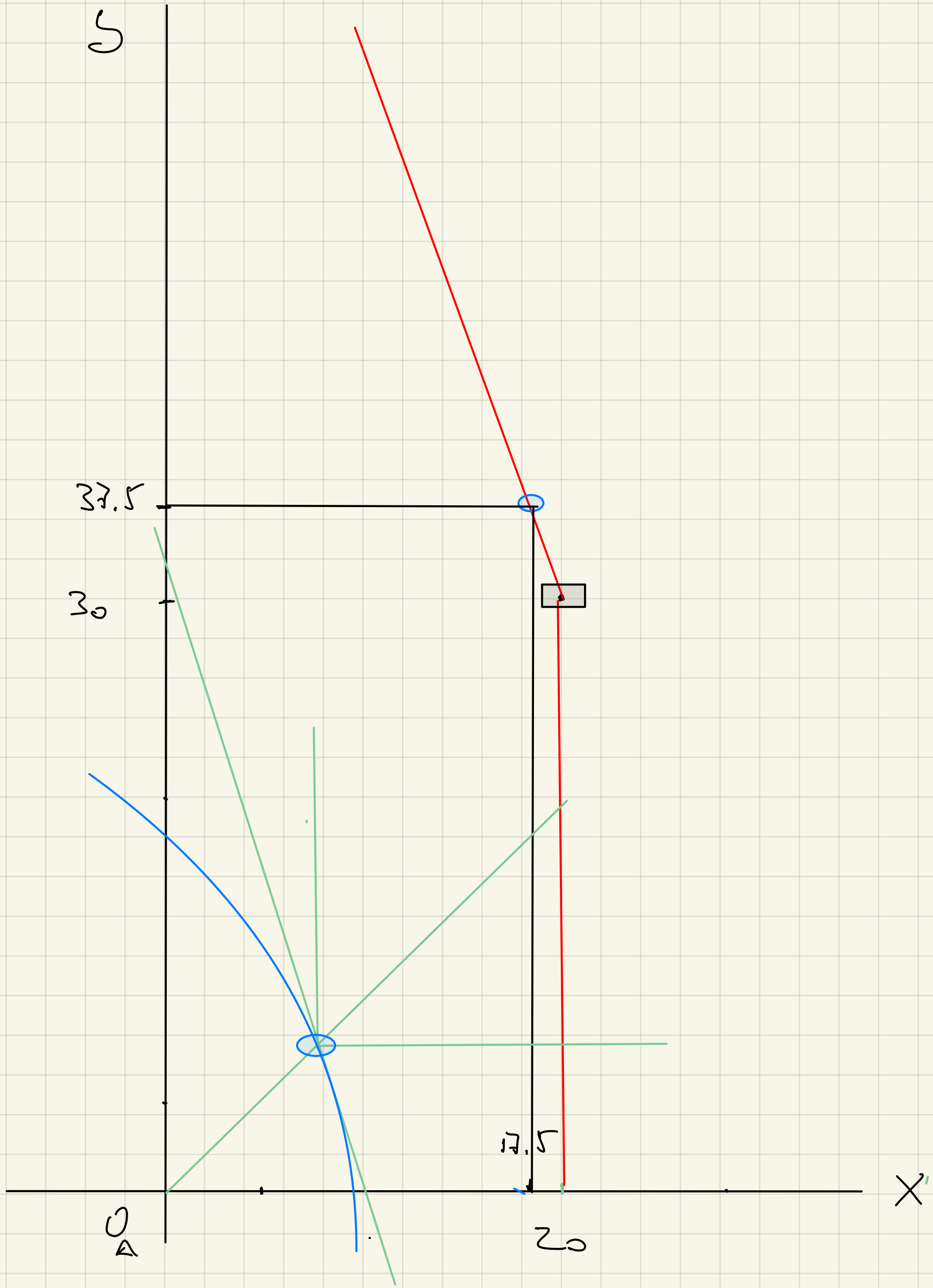
Ann's Demand of Credit: $\quad b^A(r) = \dfrac{30}{2+r}$

Bob's " " " : $\quad b^B(r) = -10$

Credit Market Clearing: $\quad b^A(r) + b^B(r) + b^F(r) = 0$

For $r > 2$: $\quad b^A(r) + b^B(r) + b^F(r) = \dfrac{30}{2+r} - 10 + 0 < \dfrac{30}{4} - 10 < 0$

Hence $r^* = 2$, and $\quad b^A(r^*) = \dfrac{30}{4} = 7.5, \quad b^B(r^*) = -10, \quad b^F(r^*) = 2.5.$



$$(x_A^*, y_B^*) = (7.5, 7.5)$$

$$(x_B^*, y_B^*) = (10, 30)$$

2. An exchange economy operates over an infinite number of consecutive dates. There is a single perishable consumption good. Every date t, N_t individuals are born. Individuals live only two dates. Thus, every date t only the consumers born at $t-1$ (the elderly), and those born at t (the young) interact. The preferences of individuals for consumption when young, x, and old, y, are represented by the utility function $u(x,y) = xy$, and their endowments are $\bar{x} = 10$ and $\bar{y} = 4$. Consider three cases regarding the evolution of the population: (i) $N_t = 2N_{t-1}$, (ii) $N_t = N_{t-1}$, and (iii) $N_t = N_{t-1}/2$, in your discussion to the questions (a) and (b) below

(a) Discuss why even if there is a credit market there is no trade is the unique competitive equilibrium. (Note that only the young can borrow or lend.) Verify that the equilibrium allocation is not Pareto optimal.

(b) Suppose now that there is a stock of money held initially by the elderly at date 1. Specifically, at date 1 each elder owns €8. Money must be accepted as a mean of exchange, i.e., every date t there is a competitive market in which the good is exchanged for money at a price of p_t (euros per unit). Write the budget constraints of a consumer. (Use the notation $\rho_t = p_t/p_{t+1}$, and note that young consumers have no money.) Calculate the set of stationary equilibrium prices (i.e., those in which ρ_t is constant over time), and identify the price ρ^* supporting the *Golden Rule.*

(a)

AGENTS' BUDGET CONSTRAINTS : $x_1 = 10 + b$, $x_2 + (1+r)b = 4$.

$$\max_{b \in \mathbb{R}} \quad (10+b)\left(4 - (1+r)b\right)$$

F.O.C. : $\quad -(1+r)(10+b) + 4 - (1+r)b = 0$

$$b(r) = \frac{4 - 10(1+r)}{2(1+r)} = \frac{2}{1+r} - 5.$$

AGENTS BORROW/LEND WHEN YOUNG. HENCE, THE YOUNG AND OLD PRESENT AT DATE t DO NOT INTERACT IN THE CREDIT MARKET.

MARKET CLEARING AT t $\quad N_t\, b(r) = 0 \iff b(r) = 0$

$$\frac{1}{1+r} = \frac{5}{2} \Rightarrow 1+r = \frac{2}{5}, \quad r^* = -\frac{3}{5}$$

..... BUT NOBODY BORROWS OR LEND.

CE ALLOCATION : $(x_1, x_2) = (10, 4)$.

THIS ALLOCATION IS NOT (TECHNICALLY) PARETO OPTIMAL!

IF THE YOUNG OF EACH GENERATION DONATED 2 UNITS OF CONSUMPTION TO THE OLD AND THESE DONATION ARE SHARED EQUALY BY THE OLD, THEN THE CONSUMPTION STREAMS ARE:

DATE:	1^{**}	2	3	---	t	---
OLD	8^{*}	8	8	---	8	---
YOUNG	8	8	8	---	8	---

(CASE $N_t = 2N_{t-1}$)

(*) $4 + 2(2)$. NOTE THAT THERE ARE TWICE AS MANY YOUNG INDIVIDUALS THAN OLD INDIVIDUALS

(**) THE CONSUMPTION STREAM OF GENERATION $t=0$ IS $(10, 8)$. ALL THE OTHER GENATIONS' CONSUMPTION STREAM IS $(8, 8)$.

THIS SCHEME IS NOT A MARKET OUTCOME ... BUT IT CAN BE THE RESULT OF A CULTURAL TRAIT BY WHICH EACH MOTHER RECIEVES 2 UNITS OF CONSUMPTION FROM EACH OF HER TWO DAUGHTERS

WHAT IF THE POPULATION IS SHINKING?

(b) MONEY (MEAN OF TRANSACTION, DEPOSIT OF VALUE)

SUPPOSE THAT THE STOCK OF MONEY IS GIVEN TO THE OLD PRESENT AT $t=1$. NATURALLY, THESE AGENTS WILL USE MONEY TO BUY AS MUCH OF THE CONSUMPTION GOOD AS THEY CAN. A YOUNG AGENT SELLING SOME OF ITS ENDOWMENT AT TIME t CAN HOLD ON THE MONEY HE RECEIVES TO BUY GOOD THE NEXT DATE. THUS, HER BUDGET CONSTRAINT IS

$$P_t (x - \bar{x}) + P_{t+1} (y - \bar{y}) = 0 \qquad (1)$$

OBVIOUSLY, IN EQUILIBRIUM $P_t > 0$, $\forall t$. HENCE (1) MAY BE WRITTEN AS

$$\frac{P_t}{P_{t+1}} (x - 10) + (y - 4) = 0$$

i.e.,

$$\rho_t x + y = 10 \rho_t + 4 \quad , \quad \rho_t = \frac{P_t}{P_{t+1}}$$

AN AGENT BORN AT t CHOOSES (x_1, x_2) TO SOLVE

$$\max \; xy$$
$$\text{s.t.} \quad p_t x + y = 10 p_t + 4$$

THE SOLUTION SOLVES THE SYSTEM

$$\left.\begin{aligned} \frac{y}{x} &= p_t \\ p_t x + y &= 10 p_t + 4 \end{aligned}\right\} \quad \begin{aligned} x_t(p_t) &= 5 + \frac{2}{p_t} \\ y_t(p_t) &= 2 + 5 p_t. \end{aligned}$$

HENCE, MARKET CLEARING REQUIRES

$$N_{t-1}\left(2 + 5p_{t-1}\right) + N_t\left(5 + \frac{2}{p_t}\right) = N_{t-1}(4) + N_t(10)$$

(i) $\boxed{N_t = 2N_{t-1}}$

$$\boxed{5p + \frac{4}{p} = 12.}$$

A STATIONARY CE (I.E., $p_t = p, \; \forall t$) SATISFIES

$$\boxed{5p^2 - 12p + 4 = 0} \iff p = \frac{12 \pm \sqrt{12^2 - 80}}{10} = \frac{12 \pm 8}{10} = \begin{cases} 2 \\ \frac{2}{5} \end{cases}$$

$p_{GR} = 2$

For $p = \frac{2}{5}$, $x\left(\frac{2}{5}\right) = 10$, $y\left(\frac{2}{5}\right) = 4$, $u\left(x\left(\frac{2}{5}\right), y\left(\frac{2}{5}\right)\right) = 40$

IN THIS CE THERE IS NO TRADE!

For $p = 2$, $x(2) = 6$, $y(2) = 12$, $u(x(2), y(2)) = 72$

PARETO SUPERIOR! THE "GOLDEN RULE" IS $p_{GR} = 2$.

(ii) $\boxed{N_{t-1} = N_t}$ MARKET CLEARING: $2 + 5p + 5 + \frac{2}{p} = 4 + 10$

i.e., $5p^2 - 7p + 2 = 0 \iff p = \frac{7 \pm \sqrt{7^2 - 40}}{10} = \begin{cases} 1 = p_{GR} \\ \frac{4}{10} \end{cases}$

$x\left(\frac{4}{10}\right) = 10$, $y\left(\frac{4}{10}\right) = 4$; $x(1) = y(1) = 7$, $u(x(1), y(1)) = 49$.

(iii) $\boxed{2N_t = N_{t-1}}$ MC: $4 + 10p + 5 + \frac{2}{p} = 2(4) + 10$;

i.e., $10p^2 - 9p + 2 = 0 \iff p^* = \frac{9 \pm \sqrt{9^2 - 80}}{20} = \begin{cases} \frac{1}{2} = p_{GR} \\ \frac{2}{5} \end{cases}$

$x\left(\frac{2}{5}\right) = 10$, $y\left(\frac{2}{5}\right) = 4$; $x\left(\frac{1}{2}\right) = \frac{9}{2}$, $y\left(\frac{1}{2}\right) = 9$, $u\left(x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right) = \frac{81}{2}$.