Exercise 1. An economy operates over two dates, today and tomorrow. The state of the economy tomorrow is uncertain: it can be in a boom (B) or in a recession (R). There is a single perishable good, consumption, and two consumers with preferences for consumption today (x), consumption tomorrow if B (y), and consumption tomorrow if R (z) represented by the utility function $u_i(x, y, z) = x + 4a_i \ln y + 8 \ln z$, where $a_1 = 1$ and $a_2 = 2$, and endowments $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (\bar{x}_2, \bar{y}_2, \bar{z}_2) = (8, 9, 6)$.

(a) (10 points) Identify the set of interior Pareto optimal allocations.

Exercise 1. (a) The set of interior Pareto optimal allocations is identified by the optimality conditions

$$\begin{aligned} MRS^1_{yx}(x_1, y_1, z_1) &= \frac{4}{y_1} = \frac{8}{y_2} = MRS^2_{yx}(x_2, y_2, z_2) \\ MRS^1_{zx}(x_1, y_1, z_1) &= \frac{8}{z_1} = \frac{8}{z_2} = MRS^2_{zx}(x_2, y_2, z_2), \end{aligned}$$

and the feasibility constraints

$$\begin{aligned} x_1 + x_2 &= \bar{x}_1 + \bar{x}_2 = 16 \\ y_1 + y_2 &= \bar{y}_1 + \bar{y}_2 = 18 \\ z_1 + z_2 &= \bar{z}_1 + \bar{z}_2 = 12. \end{aligned}$$

Thus, an interior Pareto optimal allocation satisfies $y_1 = 6$, $y_2 = 12$, $z_1 = z_2 = 6$, and $x_1 + x_2 = 12$.

(b) (30 points) Assume that there are only two markets in which agents trade today a risky security r that pays 1 unit of consumption tomorrow if the economy is in a boom and nothing if it is in a recession, and a safe security s that pays 1 unit of consumption tomorrow regardless of the state of the economy. Identify the CE allocation and security prices.

(b) Since the returns matrix

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is non-singular, this market structure is equivalent to a complete set of markets. Hence, the CE allocation is Pareto optimal and, if it is interior (which we assume for now), it satisfies $y_1^* = 6$, $y_2^* = 12$, $z_1^* = z_2^* = 6$. *Denote by* (p_x^*, p_y^*, p_z^*) *the CE Arrow-Debreu prices. Then*

$$\begin{aligned} \frac{p_y^*}{p_x^*} &= MRS_{yx}^1(x_1^*, y_1^*, z_1^*) = MRS_{yx}^2(x_2^*, y_2^*, z_2^*) = \frac{2}{3}, \\ \frac{p_z^*}{p_x^*} &= MRS_{zx}^1(x_1^*, y_1^*, z_1^*) = MRS_{zx}^2(x_2^*, y_2^*, z_2^*) = \frac{4}{3}. \end{aligned}$$

If we normalize $p_x^* = 1$, *then* $p_y^* = 2/3$ *and* $p_z^* = 4/3$. *We can then calculate the consumption today of consumers 1 and 2 as*

$$x_i^* = \bar{x}_i + 2/3(\bar{y}_i - y_i^*) + 4/3(\bar{z}_i - z_i^*) = \bar{x}_i + 2/3(\bar{y}_i - y_i^*),$$

i.e.,

$$\begin{aligned} x_1^* &= 8 + 2/3\,(9 - 6) = 10 \\ x_2^* &= 8 + 2/3\,(9 - 12) = 6. \end{aligned}$$

Now, since q_r *is the (effective) price of a unit of* y *in units of* x, *in the CE the price of this security satisfies*

$$q_r^* = \frac{p_y^*}{p_x^*} = \frac{2}{3}.$$

And since $q_s - q_r$ is the (effective) price of a unit of z in units of x, we must have

$$q_s^* - q_r^* = \frac{p_z^*}{p_x^*} = \frac{4}{3} \Leftrightarrow q_s^* = 2.$$

In order to arrive at the equilibrium allocation in the economy with security markets, the agents security trades must satisfy

$$\begin{aligned} y_i^* &= \bar{y}_i + r_i^* + s_i^* \\ z_i^* &= \bar{z}_i + s_i^*. \end{aligned}$$

Hence $r_1^* = -r_2^* = -3$, $s_1^* = s_2^* = 0$. One can readily verify that at the equilibrium prices the equations

$$x_i^* = \bar{x}_i - q_r^* r_i^* - q_s^* s_i^*$$

hold for $i \in \{1, 2\}$.

Securities: $r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $s = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Transferring consumption from 1 to (2,N)

Reduce consumption x: 1 unit.

Buy security α units

Sell " α units

$q_s \alpha = 1 + q_r \alpha$

i.e., $\alpha = \frac{1}{q_s - q_r}$

Then $\begin{pmatrix} \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \alpha - \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{q_s - q_r} \end{pmatrix}$

Thus: $\frac{p_x}{p_z} = \frac{1}{q_s - q_r}$

Alternatively, we can calculate the consumers' security demands and find the securities equilibrium prices by solving the market clearing conditions. For $i \in \{1, 2\}$ budget constraints are

$$\begin{aligned} x &\leq \bar{x}_i - q_r r - q_s s \\ y &\leq \bar{y}_i + r + s \\ z &\leq \bar{z}_i + s, \end{aligned}$$

which at the solution are binding since u_i is increasing with respect to all goods.

Substituting the value of initial endowments we can write consumer i's problem as

$$\max_{(b,z)\in\mathbb{R}^2} v_i(r,s) = (8 - q_r r - q_s s) + 4a_i \ln(9 + r + s) + 8\ln(6 + s)$$

The first order conditions for a solution to this problem are

$$\begin{aligned} \frac{\partial v_i}{\partial r} &= -q_r + \frac{4a_i}{9 + r + s} = 0 \\ \frac{\partial v_i}{\partial s} &= -q_s + \frac{4a_i}{9 + r + s} + \frac{8}{6 + s} = 0. \end{aligned}$$

Solving the system we get

$$\begin{aligned} r_i(q_r, q_s) &= \frac{4a_i}{q_r} - \frac{8}{q_s - q_r} - 3 \\ s_i(q_r, q_s) &= \frac{8}{q_s - q_r} - 6. \end{aligned}$$

Market clearing requires

$$\begin{aligned}\frac{12}{q_r} - \frac{16}{q_s - q_r} - 6 &= 0 \\ 2\left(\frac{8}{q_s - q_r} - 6\right) &= 0.\end{aligned}$$

Substituting $q_s - q_r = 4/3$ *(from the second equation) into the first equation we get* $q_r^* = 2/3$. *Hence* $q_s^* = 4/3 + q_r^* = 2$.

Hence

$$\begin{aligned}r_1(q_r^*, q_s^*) &= \frac{4}{2/3} - \frac{8}{4/3} - 3 = -3 = -r_2(q_r^*, q_s^*) \\ s_1(q_r^*, q_s^*) &= \frac{8}{4/3} - 6 = 0 = -s_2(q_r^*, q_s^*).\end{aligned}$$

Using the budget constraints we calculate the resulting allocation as

$$\begin{aligned}x_1^* &= 8 - \frac{2}{3}(-3) - \frac{4}{3}(0) = 10,\ \ x_2^* = 8 - \frac{2}{3}(3) - \frac{4}{3}(0) = 6 \\ y_1^* &= 9 - 3 + 0 = 6,\ \ y_2^* = 9 + 3 + 0 = 12 \\ z_1^* &= 6 + 0 = 6,\ \ z_2^* = 6 + 0 = 6.\end{aligned}$$

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SECURITIES: $R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$; $q_s^* = 2/3$, $q_r^* = 2$; $r_1^* = -r_2^* = -3$

$s_1^* = s_2^* = 0$

$(x_1^*, y_1^*, z_1^*) = (10, 6, 6)$

$(x_2, y_2^*, z_2^*) = (6, 12, 6)$

PO ?

$MRS^1_{xy} = \frac{y_1}{4} = \{y_1 = y_1^*\} = \frac{3}{2}$; $MRS^2_{xy} = \frac{y_2}{8} = \{y_2 = y_2^*\} = \frac{3}{2}$

$MRS^1_{xz} = \frac{z_1}{8} = \{z_1 = z_1^*\} = \frac{3}{4}$; $MRS^2_{xz} = \frac{z_2}{8} = \{z_2 = z_2^*\} = \frac{3}{4}$

Yes!

Moreover, $(p_x^*, p_y^*, p_z^*) = (1, \frac{2}{3}, \frac{4}{3})$ are Arrow-Debreu CE prices.

Indeed:

$$x = \bar{x} - q_s^* s - q_r^* r$$
$$y = \bar{y} + s + r$$
$$z = \bar{z} + s$$

$\Longleftrightarrow$

$$x = \bar{x} - q_s^* (z - \bar{z}) - q_r^* \left[y - \bar{y} - (z - \bar{z}) \right]$$

$\Longleftrightarrow$

$$x + \underset{\overset{\shortparallel}{2/3}}{q_r^*} y + (\underbrace{q_s^* - q_r^*}_{2 - 2/3}) z = \bar{x} + q_r^* \bar{y} + (q_s^* - q_r^*) \bar{z}$$

$\Longleftrightarrow$ $p_x^* x + p_y^* y + p_z^* z = p_x^* \bar{x} + p_y^* \bar{y} + p_z^* \bar{z}$

(In the Radner economy, $1/q_r$ is the rate at which consumers exchange x for y, and $1/(q_s - q_r)$ is the rate at which they exchange x for z.(*))

(*) $q_s s - q_r r = 1$, $s = r$ $\Big\}$ $s = r = \dfrac{1}{q_s - q_r}$.