Micro II: Final Exam (uc3m-Masters in Economics, May 21, 2025)

Exercise 1. Consider an economy that extends over two dates, today and tomorrow, in which there is a single perishable good, consumption. The state of nature tomorrow can be either *rainy* or *dry*. There are two consumers with identical preferences for consumption today (x), tomorrow if rainy (y), and tomorrow if dry (z), represented by the utility function $u(x, y, z) = xyz$. The economy is endowed with $\bar{x} = 8$ units of consumption today and no consumption tomorrow, i.e., $\bar{y} = \bar{z} = 0$. However, there is a firm that using x units of consumption today as input produces $2\sqrt{x}$ units of consumption tomorrow if the state is rainy and $\sqrt{x}$ if the state is dry. The property of the firm and the economy's initial endowments are equally shared by the two consumers.

(a) (20 points) Assume that there are contingent markets for all goods. Verify that the prices $(p_x^*, p_y^*, p_z^*) = (1, 1, 2)$ clear the markets, and calculate the corresponding CE allocation.

(b) (15 points) Now assume that there are no contingent markets but there is a spot market for x and markets for two securities, s and t, operating today. A unit of security s pays tomorrow 1 unit of consumption regardless of the state, while a unit of security t pays tomorrow 2 units of consumption if is rainy and nothing if it is dry. In this economy, the firm sells securities s and t and buys good x to use as input to produce the outputs required to honor its contracts, while consumers trade good x as well as securities. Denote the securities prices (given in units of consumption good today) by q_s and q_t. Calculate the competitive equilibrium prices of the securities and the corresponding allocation.

Exercise 2. A small village is considering buying an electricity generator to maintain in operation the city's street and traffic lights during electricity outages, a risk that has recently revealed significant. The cost of a generator of x megawatts (MW) of capacity is $C(x) = 200x$ thousand euros (T€). The preferences of the village's 100 residents for the generator's capacity, x, and income (in T€), y, are represented by functions $u_i(x, y) = y - (v_i - x)^2$, where $v_i = 4$ for the n residents that are merchants, and it is $v_i = 2$ for the remaining $100 - n$ residents.

(a) (10 points) Identify the Pareto optimal capacity of the generator and the Lindahl prices. (Obviously, optimal capacity depends on n.)

(b) (10 points) Determine the generator's capacity if the cost is covered by voluntary contributions.

(c) (10 points) Now assume that the city council makes the decision on the basis of the preferred capacity declared by residents, $z = (z_1, ..., z_{100})$, acquiring a generator of capacity equal to the *median of* z, and taxing residents equally to pay the cost. (The median $M(z)$ is a coordinate such that both $z_i \leq M(z)$ and $z_i \geq M(z)$ hold for at least 50 residents.) Is sincere voting a dominant strategy? What will be the generator's capacity depending on n?

Exercise 3. A risk-neutral principal bargains with an agent randomly selected from a population of agents by offering a contract. An agreement leads to a random revenue X taking values $\{12, 24, 96\}$ with probabilities $p(e)$, where $e \in \{0, 1\}$ is the effort exerted by the agent; specifically, $p(0) = (1/3, 1/3, 1/3)$ and $p(1) = (1/6, 1/6, 2/3)$, respectively. Agents' preferences are represented by the Bernoulli utility function $u(w) = \sqrt{w}$, their reservation utility is $\underline{u} = 2$, and their cost of effort is $c_H(e) = 2e$ for a fraction $q \in (0, 1)$ of agents, while for the remaining fraction it is $c_L(e) = e$.

(a) (10 points) Determine the contract the principal will offer to the agent if his type is observable and effort is verifiable.

(b) (10 points) Assume that agents' type is observable, but *effort is not verifiable.* Determine the contract the principal will offer to an agent of type H.

(c) (15 points) Now assume the principal *cannot observe the type of the agent*, but effort is verifiable. Identify the contracts the principal will offer and calculate the information rents of type L agents and the reduction of total surplus due to adverse selection. (Obviously, the answers to these questions depend on q.)

Solutions

1(a). For the given prices the firm chooses its input x solving the problem

$$\max_{x\geq 0}\ 2p_y^*\sqrt{x}+p_z^*\sqrt{x}-x=4\sqrt{x}-x,$$

The first order condition for a solution to this problem identifies the firm's input demand,

$$\frac{4}{2\sqrt{x}}-1=0\Leftrightarrow x_f^*=4.$$

Hence the firm's supply of goods y and z, and profits are

$$y_f^*=2\sqrt{4}=4,\ z_f^*=\sqrt{4}=2,\pi^*=1(4)+2(2)-4=4.$$

Thus, each consumer's budget constrain is

$$x+y+2z=\frac{1}{2}(8+4)=6.$$

In addition, a consumer demand of goods must satisfy

$$MRS_{xy}(x,y,z)=\frac{1}{p_y}\Leftrightarrow\frac{y}{x}=1\Leftrightarrow y=x$$

and

$$MRS_{xz}(x,y,z)=\frac{1}{p_z}\Leftrightarrow\frac{z}{x}=\frac{1}{2}\Leftrightarrow z=\frac{x}{2}.$$

Hence the good demands of each consumer $i\in\{1,2\}$ are

$$(x_i^*,y_i^*,z_i^*)=(2,2,1).$$

Markets clear since

$$\begin{aligned}2+2+4 &= x_1^*+x_2^*+x_f^*=\bar{x}=8\\ 2+2 &= y_1^*+y_2^*=y_f^*+\bar{y}=4+0\\ 1+1 &= z_1^*+z_2^*=z_f^*+\bar{z}=2+0.\end{aligned}$$

Hence the given price vector leads to a CE allocation, which is identified above.

1(b). The budget constrains of consumer i in this economy are

$$\begin{aligned}&(1)\ x_i + q_s s_i + q_t t_i = \tfrac{1}{2}\left(8 + \pi(q_s, q_t)\right)\\&(2)\ y_i = s_i + 2t_i\\&(3)\ z_i = s_i,\end{aligned}$$

Using equations (2) and (3) to calculate s_i and t_i *we may write the consolidated budget constraint involving consumption goods as*

$$x_i + \frac{q_t}{2} y_i + \left(q_s - \frac{q_t}{2}\right) z_i = \frac{1}{2}\left(8 + \pi(q_s, q_t)\right).$$

As shown in class, in the CE of this Radner economy the effective prices of goods x, y and z are those identified in (a), i.e.,

$$\begin{aligned}\frac{q_t^*}{2} &= p_y^* = 1\\ \left(q_s^* - \frac{q_t^*}{2}\right) &= p_z^* = 2.\end{aligned}$$

Thus,

$$(q_s^*, q_t^*) = (3, 2).$$

Let us verify that these prices would lead the firm to choose the same production activity as in part (a). The firm's problem is

$$\begin{aligned}&\max_{x,s,t}\ q_s^* s + q_t^* t - x.\\ s.t.\ :\quad & 2\sqrt{x} \geq s + 2t,\ \sqrt{x} \geq s.\end{aligned}$$

Obviously, the constraints are binding at the solution, i.e., $s = \sqrt{x}$ and $t = \sqrt{x}/2$. Thus, using the security prices calculated above we may write the firm's problem as

$$\max_{x \geq 0}\ 3\left(\sqrt{x}\right) + 2\left(\frac{\sqrt{x}}{2}\right) - x = 4\sqrt{x} - x,$$

which is the problem solved in part (a), with solution $\tilde{x}_f^* = 4$, $s_f^* = 2$, $t_f^* = 1$, and $\tilde{\pi}^* = 4$.

Since consumers 1 and 2 have the same income and buy goods at the same effective prices as in part (a), their demands are the same, i.e., $(\tilde{x}_i^*, \tilde{y}_i^*, \tilde{z}_i^*) = (2, 2, 1)$. *Hence,* $s_i^* = \tilde{z}_i^* = 1$, and $t_i^* = (\tilde{y}_i^* - s_i^*)/2 = 1/2$. *Moreover, these security prices clear the markets, since*

$$\begin{aligned}1 + 1 &= s_1^* + s_2^* = s_f^* = 2\\ \frac{1}{2} + \frac{1}{2} &= t_1^* + t_2^* = t_f^* = 1,\end{aligned}$$

Thus, these security prices lead to a CE, and generate the same allocation as in part (a).

Note. The CE identified in 1(a) can be easily calculated, showing along the way that it is indeed the unique CE of the economy. Let us go through the steps.

Normalize $p_x = 1$. The firm chooses its input x solving the problem

$$\max_{x\geq 0} \; 2p_y\sqrt{x} + p_z\sqrt{x} - x = 2A\sqrt{x} - x,$$

where

$$A := \left(p_y + \frac{p_z}{2}\right).$$

The first order condition for a solution to this problem identifies the firm's input demand,

$$\frac{A}{\sqrt{x}} - 1 = 0 \Leftrightarrow x_f\left(p_y, p_z\right) = A^2.$$

Hence the firm's supply of goods y and z are

$$y_f\left(p_y, p_z\right) = 2A, \;\; z_f\left(p_y, p_z\right) = A,$$

and its profit is

$$\pi\left(p_y, p_z\right) = 2p_yA + p_zA - A^2 = A^2.$$

Thus, each consumer's budget constrain is

$$x + p_y y + p_z z = \frac{1}{2}\left(8 + \pi\left(p_y, p_z\right)\right) = \frac{1}{2}\left(8 + A^2\right).$$

In addition, a consumer demand of goods must satisfy

$$MRS_{xy}(x, y, z) = \frac{1}{p_y} \Leftrightarrow \frac{y}{x} = \frac{1}{p_y}$$

and

$$MRS_{xz}(x, y, z) = \frac{1}{p_z} \Leftrightarrow \frac{z}{x} = \frac{1}{p_z}.$$

Hence the good demands of each consumer $i \in \{1, 2\}$ are

$$x_i\left(p_y, p_z\right) = p_y y_i\left(p_y, p_z\right) = p_z z_i\left(p_y, p_z\right) = \frac{1}{6}\left(8 + A^2\right).$$

Clearing the markets for y and z requires

$$\begin{aligned} 2A &= \frac{2}{6p_y}\left(8 + A^2\right) \qquad (y) \\ A &= \frac{2}{6p_z}\left(8 + A^2\right) \qquad (z). \end{aligned}$$

Thus, $2Ap_y = Ap_z$ implies $2p_y = p_z$ and $A = 2p_y$, and therefore market clearing for good y requires

$$2\left(2p_y\right) = \frac{2\left(8 + 4p_y^2\right)}{6p_y} \Leftrightarrow 12p_y^2 = 8 + 4p_y^2 \Leftrightarrow p_y = \pm 1.$$

Hence

$$(p_y^*, p_z^*) = (1, 2).$$

2(a). Since

$$\sum_{i=1}^{n} MRS_i(x,y) = \sum_{i=1}^{100} 2(v_i - x) = 2\left(\sum_{i=1}^{100} v_i - 100x\right)$$
$$= 2(4n + 2(100-n) - 100x)$$
$$= 200\left(2 + \frac{2n}{100} - x\right),$$

the optimal generating capacity is the solution to the equation

$$200\left(2 + \frac{2n}{100} - x\right) = 200 \Longleftrightarrow x^*(n) = 1 + \frac{n}{50}.$$

The Lindahl prices must induce all residents to choose $x^*(n)$ as the preferred capacity, i.e.,

$$MRS_i(x^*(n), y) = p_i(n) \Leftrightarrow 2(v_i - x^*(n)) = p_i(n).$$

Hence the Lindahl prices of merchants and non-merchant are

$$p_m(n) = 6 - \frac{n}{25} \text{ and } p_{nm}(n) = 2 - \frac{n}{25}.$$

(b) The utility of a resident who contributes z T€ when the other residents jointly contribute $\bar{Z} \geq 0$ T€ is

$$f_i(z, \bar{Z}) := y - z - \left(v_i - \frac{z+\bar{Z}}{200}\right)^2.$$

Taking derivatives

$$\frac{\partial f_i}{\partial z} = -1 + \frac{2}{200}\left(v_i - \frac{z+\bar{Z}}{200}\right) \leq -1 + \frac{2v_i}{200} \leq -1 + \frac{2(4)}{200} < 0.$$

That is, a single resident is not willing to contribute anything to acquire the generator because it is too expensive for a single individual. Hence under voluntary contributions the generator would not be acquired..

(c) The sincere strategy is to vote for a size z_i that solves the problem

$$\max_{x \geq 0} \; y - 2z_i - (v_i - z_i)^2$$

where $2z_i = 200z_i/100$ is the taxed paid by the resident if a generator of capacity z_i is acquired. The first order condition for a solution to this problem is

$$2(v_i - z_i) = 2 \Leftrightarrow z_i = v_i - 1.$$

To see that voting for $z_i = v_i - 1$, is a dominant strategy, let us denote by $M(z_i, Z_{-i})$ the median voter capacity. If $M(z_i, Z_{-i}) = v_i - 1$, then changing the vote cannot make the individual better off, and may make him worse off. If $M(z_i, Z_{-i}) > v_i - 1$ (respectively, $M(z_i, Z_{-i}) < v_i - 1$), then changing the vote to $\tilde{z}_i < v_i - 1$ ($\tilde{x}_i > v_i - 1$) does not change the outcome since $M(\tilde{z}_i, Z_{-i}) = M(\tilde{z}_i, Z_{-i})$, whereas changing the vote to $\tilde{z}_i > v_i - 1$ ($\tilde{x}_i < v_i - 1$) may make the resident worse off since $M(\tilde{z}_i, Z_{-i}) \geq M(\tilde{z}_i, Z_{-i})$ ($M(\tilde{z}_i, Z_{-i}) \leq M(\tilde{z}_i, Z_{-i})$). Hence, the resident cannot benefit from a deviation from her ideal size.

Therefore, in equilibrium merchant residents vote for the size $z = 3$, while non-merchant residents vote for the size $z = 1$. Therefore, if $n < 50$, a generator of capacity $M^* = 1$ is acquired, whereas if $n > 50$, a generator of capacity $M^* = 3$ is acquired.

3. (a) The expected revenues for $e = 0$ *and* $e = 1$ *are*

$$E[X(0)] = \frac{1}{3}(12 + 24 + 96) = 44,\ E[X(1)] = \frac{1}{6}(12) + \frac{1}{6}(24) + \frac{2}{3}(96) = 70.$$

Since the principal is risk-neutral and the agents are risk-averse, when effort is verifiable optimal contracts involve a fixed wage satisfying the participation constraint,

$$\sqrt{\bar{w}_i} = c_i(e) + \underline{u} \Leftrightarrow \bar{w}_i(e) = (1 + c_i(e))^2 .$$

Hence the feasible contracts the principal may offer are the contract $(0, 4)$ *to agents of both types, the contract* $(1, 9)$ *to agents of type* L *and the contract* $(1, 16)$ *to agents of type* H.

The expected profits of these contracts are

$$\begin{aligned} E[\Pi_L(0,4)] &= E[\Pi_H(0,4)] = E[X(0)] - 4 = 40 \\ E[\Pi_L(1,9)] &= E[X(1)] - 9 = 61 \\ E[\Pi_H(1,16)] &= E[X(1)] - 16 = 54. \end{aligned}$$

Therefore, the optimal contracts to offer are $(1, 16)$ *to agents of type* H *and* $(1, 9)$ *to agents of type* L*, and the expected profit of the Principal that meets randomly an high/low cost agent with probabilities* q *and* $1 - q$ *is*

$$E[\Pi(q)] = qE[\Pi_H(1,16)] + (1-q)\,E[\Pi_L(1,9)] = 54q + 61\,(1-q) = 61 - 7q.$$

(b) The contract $(e, w) = (0, 4)$ *satisfies the participation and incentive constraints and yields expected profits* $\Pi_H(0, 4) = 40$*. For agents of type* H *an optimal wage contract,* $W = (w_1, w_2, w_3)$*, involving high effort,* $e = 1$*, must satisfy the optimality equations*

$$\frac{1}{u'(w_i)} = \lambda + \mu\left(1 - \frac{p_i(0)}{p_i(1)}\right),\ i \in \{1, 2, 3\}.$$

Since $p_1(0) = p_2(0) = 1/3$ *and* $p_1(1) = p_2(1) = 1/6$ *these equations imply that* $w_1 = w_2 := w_{12}$*. Thus,* (w_{12}, w_3) *is identified by the participation and incentive constraints*

$$\frac{1}{3}\sqrt{w_{12}} + \frac{2}{3}\sqrt{w_3} = 4, \tag{PC}$$

$$\frac{1}{3}\sqrt{w_{12}} + \frac{2}{3}\sqrt{w_3} - 2 = \frac{2}{3}\sqrt{w_{12}} + \frac{1}{3}\sqrt{w_3}. \tag{IC}$$

The solution to this system is $w_{12}^* = 0$, $w_3^* = 36$*. Hence*

$$E[W^*(1)] = \frac{1}{3}(0) + \frac{2}{3}(36) = 24,$$

and the profit with this contract is

$$E[X(1)] - E[W^*(1)] = 70 - 24 = 46 > E[\Pi_H(0,4)].$$

Hence the optimal contract is $(1; W^*(1)) = (1; 0, 0, 36)$.

(c) The principal may offer the single contract $(1, 9)$*, which only the agents of type L accept, leading to expected profit*

$$E[\Pi_L(q)] = (1-q)(E[X(1)] - 9) = 61(1-q).$$

The principal may offer as well the single "pooling" contract $(1, 16)$*, involving high effort and paying a high wage, that all agents accept, leading to expected profit*

$$E[\Pi_P(q)] = (E[X(1)] - 16) = 54.$$

Alternatively, the principal may offer an incentive compatible menu involving high effort for the type L *and low effort for the type* H*. The PC of the type* H *identifies the contract* $(e_H, \varpi_H) = (0, 4)$*. The IC of type* L *requires* $(\tilde{e}_L, \tilde{w}_L) = (1, \tilde{w}_L)$ *satisfy*

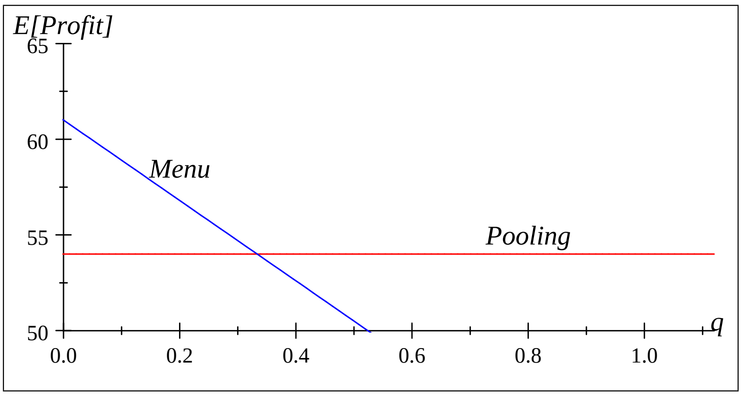
$$\sqrt{\tilde{w}_L} - 1 = \sqrt{\varpi_H} \Leftrightarrow \tilde{w}_L = (2+1)^2 = 9.$$

For this menu of contracts the expected profit is

$$E[\Pi_M(q)] = 40q + 61(1-q) = 61 - 21q > E[\Pi_L(q)],$$

i.e., the menu dominates offering the single contract $(1, 9)$*.*

The graphs of the functions $E[\Pi_P(q)]$ *and* $E[\Pi_M(q)]$*, displayed below, show that for* $q \leq 1/3$ *the menu is optimal, and* $q \geq 1/3$ *the pooling contract* $(1, 16)$ *is optimal.*



In the absence of adverse selection the principal captures the entire surplus, given for $q \in (0, 1)$ *by* $E[\Pi(q)] = 61 - 7q$*, as calculated in part (a).*

With adverse selection, for $q \in (0, 1/3)$ *the principal offers the menu, which entails no surplus for either type of agent, and therefore the total surplus is just the principal's profit, which is*

$$E[\tilde{\Pi}(q)] = 61 - 21q.$$

Hence, the surplus loss due to adverse selection is

$$E[\Pi(q)] - E[\tilde{\Pi}(q)] = 61 - 7q - (61 - 21q) = 14q > 0.$$

For $q \in (0, 1/3)$*, however, the principal offers the pooling contract* $(1, 16)$*. Thus, the optimal effort is exerted by both high and low cost agents, and therefore no surplus is lost. This can be easily verified: Type* L *agents capture a monetary surplus of* $c_H(1) - c_L(1) = 16 - 9 = 7$*, while the principals profit is* $E[\Pi_P(q)] = 54$*. Hence, the surplus is*

$$Surplus = E[\tilde{\Pi}(q)] = 54 + 7(1-q) = 61 - 7q = E[\Pi(q)],$$

where $E[\Pi(q)]$ *is the surplus with complete information calculated in part (a).*