Microeconomics II: Final Exam (May 22, 2024)

Exercise 1. A pure exchange economy extends over two periods, today and tomorrow, and the state of nature tomorrow can either be either *cheerful* or *sad*. There is a single perishable good, consumption, and two consumers whose preferences over consumption today (x), consumption tomorrow if cheerful (y), and consumption tomorrow if sad (z) are represented by the utility functions $u_1(x,y,z) = xz$ and $u_2(x,y,z) = xy^2$, and whose endowments are $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (4,4,0)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = (6,0,6)$, respectively.

(a) (20 points) Describe consumers' budget constraints if there are contingent markets for all commodities, and calculate the competitive equilibrium prices and allocations. (Denote the prices by (p_x, p_y, p_z), and normalize $p_x = 1$.)

(b) (15 points) Describe consumers' budget constraints if there are no contingent markets, but there is a credit market and a market for a security that pays 1 unit of consumption tomorrow if cheerful and 3 units of consumption tomorrow if sad. Then calculate the competitive equilibrium security price q^* and interest rate r^*. (Normalize the spot prices $\hat{p}_x = \hat{p}_y = \hat{p}_z = 1$ and denote by b_i and s_i consumer i's demands of credit and security. By consolidating a consumer's budget constraints into a single equation involving consumption goods you will be able to relate (q^*, r^*) with equilibrium prices of part (a).)

Exercise 2. Robinson (R) and Friday (F) are the only inhabitants of a small island. They care about weakly fish consumption (x) and days of leisure (y), and they have identical preferences, represented by the utility function $u(x,y) = x + y$. Their weakly catch of fish depends on the number of days they both fish (they crowd out each other at the fishing spot), and is given for $i \in \{R, F\}$ by $z_i\,(7 - z_R - z_F)$.

(a) (10 points) Determine the socially optimal number of days a week the two men should fish, and the set of Pareto optimal allocations.

(a) (10 points) Calculate how many days a week they will fish if they choose individually.

(c) (10 points) Friday, unhappy with the status quo (b), makes a *take it or leave offer* to Robinson: *give up fishing and I will compensate you with T units of fish.* Which is the lowest value of T that Robinson would accept? Which is the largest value of T that Friday would be willing to offer? If they reach an agreement, would it lead to a Pareto optimal allocation?

Exercise 3. The revenue of a risk-neutral principal, X, is a random variable that takes the values $\{3,4,8\}$ with probabilities that depend on the level of effort exerted by an agent, $e \in \{1/4, 1\}$, and are given by $p(1/4) = (1,0,0)$, and $p(1) = (1/3, 1/3, 1/3)$, respectively. There is a population of agents whose preferences are represented by the Bernoulli utility function $u(w) = \sqrt{w}$, and whose reservation utility is $\underline{u} = 0$. For a fraction $q \in (0,1)$ of the agents the cost of effort is $c_H(e) = e$, while for the remaining fraction it is $c_L(e) = e/2$.

(a) (10 points) Assume that agents' type is observable, and effort is verifiable. Determine the contract the principal will offer to each type of agent.

(b) (10 points) Assume that the agents' type is observable, but *effort is not verifiable.* Determine the contract the principal will offer to type L agents.

(c) (15 points) Now assume that effort is verifiable, but the principal *does not observe the agents' type*. Determine the menu contracts the principal will offer for $q \in (0,1)$.

Solutions

Exercise 1.

(a) The budget constrain of consumer $i \in \{1,2\}$ is

$$x - \bar{x}_i + p_y(y - \bar{y}_i) + p_z(z - \bar{z}_i) \leq 0.$$

For $(p_y, p_z) \gg 0$, consumers' demands are

$$\begin{aligned} x_1(p_y, p_z) &= 2(1+p_y), \; y_1(p_y,p_z) = 0, \; z_1(p_y,p_z) = \frac{2}{p_z}(1+p_y) \\ x_2(p_y, p_z) &= 2(1+p_z), \; y_2(p_y,p_z) = \frac{4}{p_y}(1+p_z), \; z_2(p_y,p_z) = 0. \end{aligned}$$

Solving the system of market clearing conditions

$$\begin{aligned} y_1(p_y,p_z) + y_1(p_y,p_z) &= \bar{y}_1 + \bar{y}_2 \Leftrightarrow \frac{4}{p_y}(1+p_z) = 4 \\ z_1(p_y,p_z) + z_2(p_y,p_z) &= \bar{z}_1 + \bar{z}_2 \Leftrightarrow \frac{2}{p_z}(1+p_y) = 6, \end{aligned}$$

we get $(p_y^*, p_z^*) = (2,1)$. The equilibrium allocation is

$$(x_1^*, y_1^*, z_1^*) = (6,0,6), \; (x_1^*, y_1^*, z_1^*) = (4,4,0).$$

(b) The budget constraints of a consumer are (I have suppressed the subindices to avoid notation):

$$\begin{aligned} x + qs &\leq \bar{x} + b, \\ y &\leq \bar{y} - (1+r)b + s, \\ z &\leq \bar{z} - (1+r)b + 3s. \end{aligned}$$

Clearly, the budget constraints are binding at the solution. Solving for b and s we may rewrite the consumer's (single) budget constraint involving consumption as

$$x - \bar{x} + (\frac{3}{2(1+r)} - \frac{q}{2})(y - \bar{y}) + (\frac{q}{2} - \frac{1}{2(1+r)})(z - \bar{z}) \leq 0.$$

Hence a consumer's problem is identical to that of part (a). The equilibrium values of q and r can be obtained by solving the system of equations

$$\begin{aligned} \frac{3}{2(1+r)} - \frac{q}{2} &= p_y^* \\ \frac{q}{2} - \frac{1}{2(1+r)} &= p_z^*, \end{aligned}$$

which yields $(q^*, r^*) = (5, -2/3)$. Of course, the equilibrium allocation is that of part (a).

Exercise 2.

(a) The number of days both men fish, $z \in [0, 14]$*, that maximizes social welfare is the solution to the problem*

$$\max_{z\in[0,14]} z(7-z) + 14 - z,$$

which is $z^* = 3$*. The total catch of fish is*

$$z^*(7-z^*) = 12,$$

and the number of days left for leisure activities is

$$14 - z^* = 11.$$

Hence, the set of Pareto optimal allocations is

$$P = \{[(x_R, y_R), (x_F, y_F)] \mid x_R + x_F = 12,\ y_R + y_F = 11\}.$$

Thus, for any Pareto optimal allocation

$$u_R(x_R, y_R) + u_F(x_F, y_F) = x_R + y_R + x_F + y_F = 23.$$

(b) In order to choose how many days to fish $i \in \{R, F\}$ *solves the problem*

$$\max_{z_i\in[0,7]} z_i(7 - z_i - z_j) + 7 - z_i$$

The F.O.C. for an interior solution yields

$$z_i = 3 - \frac{z_j}{2}.$$

Hence, in equilibrium $z_1 = z_2 = z_v$*, where*

$$z_v = 3 - \frac{z_v}{2} = 2.$$

Thus, each men fishes 2 *days a week for a total catch of fish equal to* 6*, and enjoys* 5 *days of leisure for a total utility equal to* $u_i(6+5) = 11$*,* $i \in \{R, F\}$*. Thus, this allocation is not Pareto optimal.*

(c) If Robinson accepts to give up fishing in exchange for a compensation T*, then Friday would choose the number of days he fishes* z *by solving the problem*

$$\max_{z}\ z(7-z) + 7 - z - T$$

whose solution is obviously $z^* = 3$*, leading to a fish catch of* 12 *units. Thus, Friday's and Robinson's utilities would be*

$$u_F(12 - T, 7 - 3) = 16 - T,\ u_R(T, 7) = T + 7.$$

Hence the sum of their utilities is 23*, and therefore the allocation is Pareto optimal.*

The necessary conditions for an offer to be acceptable to both Friday and Robinson are

$$16 - T \geq 11, \text{ and } T + 7 \geq 11,$$

i.e.,

$$T \in [4, 5].$$

Exercise 3.

(a) The expected revenues for $e = 1/4$ *and* $e = 1$ *are*

$$E[X(1/4)] = (1)\,3 + (0)\,(4+8) = 3,\;\; E[X(1)] = \frac{1}{3}(3+4+8) = 5.$$

Since the principal is risk-neutral and the agents are risk-averse, when effort is verifiable optimal contracts involve a fixed wage satisfying the participation constraint,

$$\sqrt{\bar{w}_i} = c_i(e) + \underline{u} \Leftrightarrow \bar{w}_i(e) = c_i(e)^2.$$

Hence the principal may offer either the contracts $(e_H, \bar{w}_H) = (1/4, 1/16)$ *or* $(e_H, \bar{w}_H) = (1,1)$ *to agents of type* H*, and the contracts* $(e_L, \bar{w}_L) = (1/4, 1/64)$ *or* $(e_L, \bar{w}_L) = (1, 1/4)$ *to agents of type* L.

For agents of type H *the expected profits of these contracts are*

$$E[X(1/4)] - \bar{w}_H\,(1/4) = 3 - \frac{1}{16} \simeq 2.94,\;\; E[X(1)] - \bar{w}_H\,(1) = 5 - 1 = 4.$$

Therefore, the optimal contract to offer agents of type H *is* $(e_H^*, \bar{w}_H^*) = (1,1)$.

For agents of type L *the expected profits of these contracts are*

$$E[X(1/4)] - \bar{w}_L\,(1/4) = 3 - \frac{1}{64} \simeq 2.99,\;\; E[X(1)] - \bar{w}_L\,(1) = 5 - \frac{1}{4} = 4.75.$$

Therefore, the optimal contract to offer agents of type L *is* $(e_L^*, \bar{w}_L^*) = (1, 1/4)$.

(b) For agents of type L *the contract* $(e_L, \bar{w}_L) = (1/4, 1/16)$ *satisfies the participation and incentive constraints and yield profits*

$$E[X(1/4)] - \bar{w}_L\,(1/4) = 2.99,$$

as shown above. The optimal wage contract $W = (w_1, w_2, w_3)$ *involving high effort,* $e = 1$*, must satisfy the optimality equations*

$$\frac{1}{u'\,(w_i)} = \lambda + \mu\left(1 - \frac{p_i(1/4)}{p_i(1)}\right),\;\; i \in \{1,2,3\}.$$

Since $p_2(1/4) = p_3(1/4) = 0$ *and* $p_2(1) = p_3(1) = 1/3$ *these equations imply that* $w_2 = w_3 := w_{23}$*. Thus,* (w_1, w_{23}) *is identified by the participation and incentive constraints*

$$\frac{1}{3}\sqrt{w_1} + \frac{2}{3}\sqrt{w_{23}} = \frac{1}{2}, \tag{PC}$$

$$\frac{1}{3}\sqrt{w_1} + \frac{2}{3}\sqrt{w_{23}} - \frac{1}{2} = \sqrt{w_1} - \frac{1}{8}. \tag{IC}$$

The solution to this system is $w_1^* = 1/64$, $w_{23}^* = 121/256$*. Hence* $E[W^*(1)] = (w_1^* + 2w_{23}^*)\,/3 = 41/128$*, and the profit with this contract is*

$$E[X(1)] - E[W^*(1)] = 5 - \frac{41}{128} \simeq 4.68 > 2.99.$$

Hence the optimal contract is $(1, W^*)$.

(c) The principal may offer the single contract $(e_L^*, \bar{w}_L^*) = (1, 1/4)$*, which only the agents of type L accept, leading to the an expected profit of*

$$\Pi_L(q) = (1-q)(E[X(1)] - 1/4) = 4.75(1-q),$$

The principal may offer as well the pooling contract $(e_H^*, \bar{w}_H^*) = (1, 1)$*, involving high effort and paying a high wage, that all agents accept, leading to profits*

$$\Pi_H(q) = (E[X(1)] - 1) = 4.$$

Alternatively, the principal may offer an incentive compatible menu. Such menu should involve high effort for the type L and low effort for the type H. The PC of the type H identifies the contract $(\tilde{e}_H, \tilde{w}_H) = (e_H, \bar{w}_H) = (1/4, 1/16)$*. The IC of type L requires* $(\tilde{e}_L, \tilde{w}_L) = (1, \tilde{w}_L)$ *satisfy*

$$\sqrt{\tilde{w}_L} - \frac{1}{2} = \sqrt{\tilde{w}_H} - \frac{1}{8}. \tag{IC_L}$$

Hence

$$\tilde{w}_L = \left(\frac{1}{4} - \frac{1}{8} + \frac{1}{2}\right)^2 = \frac{25}{64}$$

For this menu of contracts the expected profit is

$$\begin{aligned}\Pi_M(q) &= q\left(E[X(\frac{1}{4})] - \frac{1}{16}\right) + (1-q)\left(E[X(1)] - \frac{25}{64}\right) \\ &= \frac{295 - 107q}{64}\end{aligned}$$

The graphs of the functions $\Pi_M(q)$, $\Pi_M(q)$ *and* $\Pi_M(q)$*, displayed below, show that for low values of* $q \leq 9/197$ *the single contract* $(e_L^*, \bar{w}_L^*)$ *is optimal and for large value of* $q \geq 39/107$ *the pooling contract* $(e_H^*, \bar{w}_H^*)$ *is optimal, while for intermediate values of* $q \in [9/197, 39/107]$ *the menu of contracts is optimal.*