1. Consider the contract design problem of a risk neutral principal whose revenue depends on the effort exerted by an agent, $e \in \{0, 1\}$, as described in the following table:

X	0	10	25
$p(0)$	$\frac{1}{2}$	$\frac{1}{2}$	0
$p(1)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

The reservation utility of this agent is $\underline{u} = 1$, his cost of effort is $c(e) = e$ and his von Neumann-Morgenstern utility function is $u(w) = \sqrt{w}$. Determine the optimal contract when effort level is observed and when it is not. What is the impact of moral hazard on the welfare of the agent and on the social surplus?

In the absence of moral hazard:

$$E[X(0)] = \frac{1}{2}(0+10) + 0(25) = 5$$

$$E[X(1)] = \frac{1}{4}(0+10) + \frac{1}{2}(25) = 15$$

Also: $\sqrt{w} = C(0) + \underline{u} = 1 \Rightarrow \overline{w}(0) = 1$

$$\sqrt{w} = c(1) + \underline{u} = 2 \Rightarrow \overline{w}(1) = 4$$

Hence the optimal contract is

$$[e^*, \overline{w}(e^*)] = [1; 4],$$

and the expected profit is $E[X(1)] - \overline{w}(1) = 11$ //

With moral hazard:

Since $l_i = \frac{P_i(0)}{P_i(1)} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$ for $i = 1,2 \Rightarrow w_1^* = w_2^* := x^2$.

Thus, the PC and IC equations are:

$$(PC) \qquad \frac{1}{4}(2x) + \frac{1}{2}(y) = 2$$

$$(IC) \qquad \frac{1}{4}(2x) + \frac{1}{2}(y) - 1 = \frac{1}{2}(2x),$$

where $y^2 = w_3$. The solution to this system is $x = 1, y = 3$.

Hence the optimal contract inducing the Agent to exert effort $e = 1$ is $w_1^* = w_2^* = 1$, $w_3 = y^2 = 9$. Hence the expected wage is

$$E[W^*(1)] = \frac{1}{4}(1+1) + \frac{1}{2}(9) = \underline{\underline{5}}$$

The Principal's expected profit is $E[X(1)] - E[W^*(1)] = 15 - 5 = \underline{\underline{10}}$

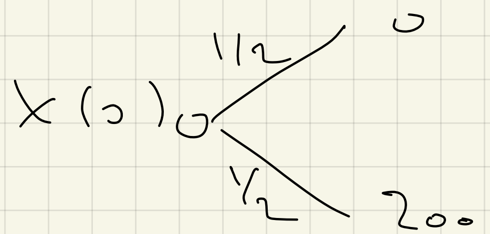
The lost surplus is $E[W^*(1)] - \overline{W}(1) = 5 - 4 = 1.$

2. A salesman's probability of selling an encyclopedia is $p \in (1/2, 1)$ if he exerts effort, and 1/2 otherwise. His cost of exerting effort is 20, and his reservation utility is $\underline{u} = 50$. A sale generates a revenue of 200 euros to the company. Both the salesman and his company are risk-neutral.

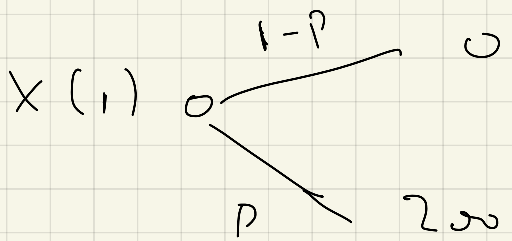
(a) The company cannot monitor the salesman, but can pay him wages contingent on success. Write down the company's contract design problem. Calculate the optimal contract for each $p \in (1/2, 1)$. Is the optimal contract socially optimal? Explain this result.

(b) Now suppose the salesman is risk averse, and his von Neumann-Morgenstern utility function is $u(w) = \cancel{20 \ln w}\ \frac{\ln w}{20}$. Determine the optimal contract for $p = 4/5$.

We have:



$; \quad E[X(0)] = \frac{1}{2}(0 + 200) = 100$



$; \quad E[X(1)] = p(200) + (1-p)(0) = 200p.$

In the absence of moral hazard, $\bar{w}(0) = 50$, $\bar{w}(1) = 70$ are the optimal fixed wage to pay for the agent to either exert no effort or to exert effort, respectively. Hence

$$E[X(1)] - \overline{w}(1) \geq E[X(0)] - \overline{w}(0) \iff 200p - 70 \geq 100 - 50$$

$$\iff p \geq \frac{120}{200} = \frac{3}{5}.$$

With moral hazard the optimal wage contract for $\underline{e=1}$ solves the system

$$(PC) \quad p w_1 + (1-p) w_2 = 70$$

$$(IC) \quad p w_1 + (1-p) w_2 - 20 = \frac{1}{2}(w_1 + w_2)$$

Hence $w_1^*(p) = \dfrac{100p - 30}{2p-1}$, $w_2^*(p) = \dfrac{100p - 70}{2p-1}$

and $E[w^*(p)] = \underline{\underline{70}} = \overline{w}(1)$.

Moral hazard does not affect the expected wage because the agent is risk-neutral!

Limited Liability (LL) If $100p - 70 < 0$, i.e., $\underline{\underline{p < 7/10}}$, then LL implies $\tilde{w}_2 = 0$ and hence

$$(PC) \quad p\tilde{w}_1 \geq 70 \implies \tilde{w}_1 \geq 70/p$$

$$(IC) \quad p\tilde{w}_1 - 20 \geq \frac{\tilde{w}_1}{2} \implies w(p - \tfrac{1}{2}) \geq 20$$

If we set $\tilde{w}_1 = \frac{70}{p}$, then for the IC constraint to hold it is required that

$$p\left(\frac{70}{p}\right) - 20 \geq \frac{1}{2}\left(\frac{70}{p}\right) \iff p \geq 7/10,$$

which we are assuming does not hold.

Hence for $p < 7/10$ the (PC) is not binding, while the IC is; hence $\tilde{w}_1$ solves

$$p\tilde{w}_1 - 20 = \frac{1}{2}(\tilde{w}_1)$$

i.e.,

$$\tilde{w}_1 = \frac{40}{2p-1} > \frac{70}{p} \quad \text{for } p < 7/10. \quad \text{(Check!)}$$

Hence LL works in favor of the Agent, allowing him to capture some surplus.

(b)

For $e=0$, the fixed wage contract $(0, \bar{w}(0))$, where $\bar{w}(0)$ is the solution to the equation

$$u(w) = 20 \ln w = 50 \iff \bar{w}(0) = e^{5/2} \simeq 12.18,$$

is incentive compatible, and

$$E[X(0)] - \bar{w}(0) = 100 - 12.18 < 100$$

For $e=1$, the fixed wage contract is $(1, \bar{w}(1))$ where

$$u(w) = 20 \ln w = 70 \iff \bar{w}(1) = e^{7/2} \simeq 33.11,$$

and

$$E[X(1)] - \bar{w}(1) = 200p - \bar{w}(1) = 200\left(\frac{4}{5}\right) - 33.11 = 126.88$$

Hence, in the absence of moral hazard the optimal contract is $(1, \bar{w}(1))$.

With moral hazard the contract $(0, \bar{w}(0))$ remains incentive compatible, but for $e=1$ the optimal contract is the random wage (w_S, w_{NS}) solving the system:

(PC) $\quad \frac{4}{5}(20 \ln w_S) + \frac{1}{5}(20 \ln w_{NS}) = 70$

(IC) $\quad \frac{4}{5}(20 \ln w_S) + \frac{1}{5}(20 \ln w_{NS}) - 20 = \frac{1}{2}(20 \ln w_S + 20 \ln w_{NS})$

Hence $w_S = e^{25/6} \simeq 76.2$; $w_{NS} = 2.3$. The expected profits with this contract is

$$\frac{4}{5}\left(200 - e^{25/6}\right) + \frac{1}{5}\left(0 - e^{5/6}\right) \simeq 107.94 > 100$$

Hence this contract is optimal.

3. An individual's preferences are represented by the von Neumann-Morgenstern utility function $u(w) = \sqrt{w}$, and his wealth is $W = 100$ euros. The individual faces the risk of suffering a loss of $L = 36$ euros. He may take a precautionary action, whose cost is $c = 4/10$, that would reduce the probability of suffering this loss from $p_0 = 3/4$ to $p_1 = 1/2$. In addition, the individual may subscribe an insurance contract (I, D), where I is the premium, in euros, and $D \in [0, 36]$ is the deductible (that is, the part of the loss no covered by the policy). Determine the insurance policy that a monopolistic risk neutral insurance company will offer when the precautionary action is contractible, and when it is not.

In the <u>absence</u> of insurance, the individual will take the precautionary action, i.e., $e = 1$, since

$$Eu(e=1) = \frac{1}{2}\sqrt{W} + \frac{1}{2}\sqrt{W-L} - C(1) = \frac{1}{2}\sqrt{100} + \frac{1}{2}\sqrt{100-36} - \frac{4}{10}$$

$$= 8.6$$

$$> \frac{3}{4}\sqrt{W} - \frac{1}{4}\sqrt{W-L} - C(0) = \frac{3}{4}\sqrt{100} + \frac{1}{4}\sqrt{100-36} - 0$$

$$= 8.5$$

If the action is verifiable, the maximum premium the individual is willing to pay for full the insurance involving taking the precautionary action is the solution to the equation

$$u(W-I) = 8.6 \iff \sqrt{W-I} = 8.6$$

$$\iff I = W - 8.6^2 = 26.02.$$

If the action is not verifiable, then the monopoly must provide incentive by offering a policy with a deductible $D > 0$, to satisfy both the PC and IC constraints, i.e.,

$$PC: \quad \frac{1}{2}\sqrt{100-I} + \frac{1}{2}\sqrt{100-I-D} - c(1) = 8.6$$

$$IC: \quad \frac{1}{2}\sqrt{100-I} + \frac{1}{2}\sqrt{100-I-D} - c(1) = \frac{1}{4}\sqrt{100-I} + \frac{3}{4}\sqrt{100-I-D}.$$

The solution to this system is

$$I^* = \frac{99}{25} = 3.96$$

$$D = \frac{144}{5} = 28.8.$$

The monopoly expected profit is $I - \frac{1}{2}(L-D) = 3.96 - \frac{1}{2}(36-28.8) = \underline{0.36}$