Exercise 2. Consider a market for used cars whose qualities, indexed by the sellers' cost, are uniformly distributed in the interval $[2, 6]$. Buyers are risk-neutral and value each quality 20% more than sellers. Naturally, each seller knows the quality of the good he sells, but quality is not observable to buyers prior to purchase. Assume that there are more buyers than sellers.

(a) Determine the market supply and the average quality of the cars offered at each price.

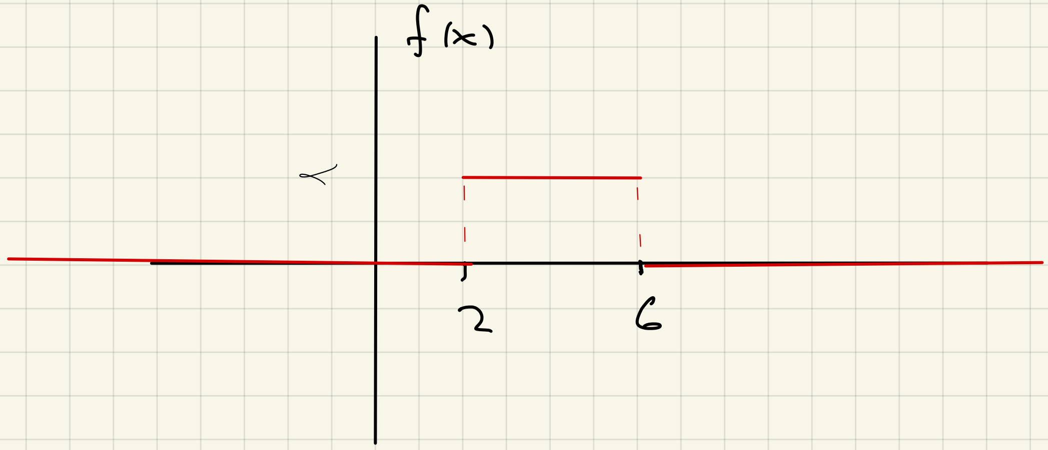
(b) Calculate the market equilibrium.

SUPPLY

$$
s(p) = \begin{cases} 0 & p < 2 \\ p-2 & p \in [2,6] \\ 4 & p > 6. \end{cases}
$$

DEMAND : MORE INVOLVED ----

$$X \sim U[2,6]$$



THUS, THE PDF OF X IS, FOR $x \in \mathbb{R}$,

$$f_X(x) = \begin{cases} \alpha & x \in [2,6] \\ 0 & x \notin [0,6], \end{cases}$$

WHERE α SATISFIES

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_2^6 \alpha\,dx = \alpha\,[x]_2^6 = 4\alpha \iff \alpha = \frac{1}{4}.$$

AND THE CDF OF X IS, FOR $x \in \mathbb{R}$,

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz = \begin{cases} 0 & \text{IF } x < 2 \\ \displaystyle\int_2^x \frac{dz}{4} = \frac{[z]_2^x}{4} = \frac{x-2}{4} & \text{IF } x \in [2,6] \\ 1 & \text{IF } x > 6. \end{cases}$$

LET $p \in [2, 6]$. WHAT IS THE DISTRIBUTION OF $X / X \le p$?

BAYES' RULE: LET (Ω, Pr) BE A PROBABILITY MODEL, AND LET A AND B ANY TWO EVENTS, $A, B \subset \Omega$, WHERE $Pr(B) > 0$. THEN

$$Pr(A / B) = \frac{Pr(A \cap B)}{Pr(B)}.$$

LET US APPLY BAYES' RULE TO CALCULATE $f_{X/X \le p}$.

LET $p \in (2, 6)$, AND $x \in \mathbb{R}$:

$$F_{X|X\le p}(x) = Pr(X \le x / X \le p) = \frac{Pr(X \le x, X \le p)}{Pr(X \le p)}$$

SINCE

$$Pr(X \le x, X \le p) = \begin{cases} F_X(x) & \text{IF } x < p \\ F_X(p) & \text{IF } x \ge p, \end{cases}$$

THEN

$$F_{X/X\le p}(x) = \begin{cases} \dfrac{F_X(x)}{F_X(p)} & \text{if } x \le p \\[2ex] \dfrac{F_X(p)}{F_X(p)} & \text{if } x \le p \end{cases} = \begin{cases} 0 & \text{if } x < 2 \\ \dfrac{x-2}{p-2} & \text{if } x \in [2, p] \\ 1 & \text{if } x > p. \end{cases}$$

HENCE

$$f_{X/X\le p}(x) = F'_{X/X\le p}(x) = \begin{cases} 0 & \text{if } x < 2 \\ \dfrac{1}{p-2} & \text{if } x \in [0, p] \\ 0 & \text{if } x > p, \end{cases}$$

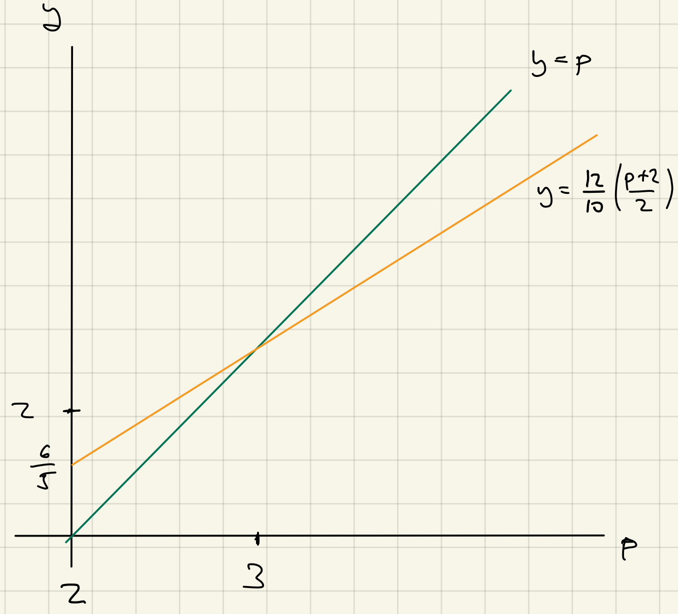
AND

$$E[X / X \le p] = \int_{-\infty}^{\infty} x f_{X/X\le p}(x)\, dx = \int_2^p \frac{x}{p-2}\, dx = \frac{\left[\frac{x^2}{2}\right]_2^p}{p-2}$$

$$= \frac{1}{2} \frac{p^2-4}{p-2} = \frac{1}{2} \frac{\cancel{(p-2)}(p+2)}{\cancel{(p-2)}} = \frac{p+2}{2}.$$

A BUYER WILL BY AT A PRICE $p \in [2, 6]$ iff

$$\frac{12}{10} E[X \mid X \le p] = \frac{12}{10}\left(\frac{p+2}{2}\right) \ge p \iff p \le 3$$



HENCE DEMAND FOR $p \in \mathbb{R}_+$ IS

$$D(p) = \begin{cases} n_B & \text{IF } p < 3 \\ [0, n_B] & \text{IF } p = 3 \\ 0 & \text{IF } p > 3. \end{cases}$$

AND THE CE PRICE IS $p^* = 3$, AND ONLY THE QUALITIES IN THE INTERVAL $[2, 3]$ TRADE.