The Agency Problem: Effort Choice

Example 2. Assume that the set of feasible effort levels is $[0, 1]$, and that the cost of effort is $c(e) = e/2$. Revenue X takes two values $x_1 = 4$, and $x_2 = 8$ with probabilities $p(e)$ and $1 - p(e)$, where $p(e) = (2 - e)/3$. Identify the optimal contract assuming that $\underline{u} = 1$, and

(a) $\pi(z) = z$ and $u(w) = \sqrt{w}$,

(b) $\pi(z) = \sqrt{z}$ and $u(w) = \beta w$,

(c) $\pi(z) = \sqrt{z}$ and $u(w) = \sqrt{w}$.

SOLUTION

$$\mathbb{E}\{X(e)\} = p(e)\, x_1 + (1-p(e))\, x_2 \iff \boxed{\mathbb{E}\{X(e)\} = \frac{1}{3}(16+4e)}$$

(a) THE OPTIMAL WAGE CONTRACT IS, IN THIS CASE, A FIX WAGE $\bar{w}(e)$, WHERE $\bar{w}(e)$ SOLVES THE PC EQUATION

$$u(w) = c(e) + \underline{u} \iff \sqrt{w} = \frac{e}{2} + 1 \iff \boxed{\bar{w}(e) = \left(\frac{e}{2}+1\right)^2}$$

HENCE, THE OPTIMAL EFFORT SOLVES

$$\max_{e\in[0,1]} \frac{1}{3}(16+4e) - \left(\frac{e}{2}+1\right)^2$$

THAT IS, e^* SOLVES

$$\frac{4}{3} - \frac{e}{2} - 1 = 0 \iff \boxed{e^* = \frac{2}{3}} \Rightarrow \bar{w}(e^*) =$$

THE OPTIMAL CONTRACT IS $\underline{(e^*, w^*) = \left(\frac{2}{3}, \left(\frac{16}{9}, \frac{16}{9}\right)\right)}$.

(b) IN THIS CASE THE OPTIMAL WAGE CONTRACT IS A FRANCHISE,

AND THE LEVEL OF EFFOR SOLVES THIS PROBLEM

$$\max_{e \in [0,1]} \quad y(e) = \mathbb{E}\{X(e)\} - \frac{C(e) + u}{\beta} = \frac{1}{3}(16 + 4e) - \frac{\left(\frac{e}{2} + 1\right)}{\beta}.$$

$$= \left(\frac{16}{3} - \frac{1}{\beta}\right) + \left(\frac{4}{3} - \frac{1}{2\beta}\right) e$$

HENCE

$$(e^*, y(e^*)) = \begin{cases} \left(0, \frac{16}{3} - \frac{1}{\beta}\right) & \text{IF } \beta < \frac{3}{8} \\ \left(e, \frac{8}{3}\right) \ \forall e \in [0,1] & \text{IF } \beta = \frac{3}{8} \\ \left(1, \frac{20}{3} - \frac{3}{2\beta}\right) & \text{IF } \beta > \frac{3}{8}. \end{cases}$$

(c) THE OPTIMAL WAGE CONTRACT $W=(w_1,w_2)$ IS THE SOLUTION TO THE SYSTEM

$$
\left.\begin{aligned}
&E u(W) = c(e) + \underline{u} &&\iff&& p(e)\sqrt{w_1} + (1-p(e))\sqrt{w_2} = \frac{e}{2}+1 \\
&\frac{\pi'(x_1-w_1)}{\pi'(x_2-w_2)} = \frac{u'(w_1)}{u'(w_2)} &&\iff&& \frac{x_2-w_2}{x_1-w_2} = \frac{\sqrt{w_1}}{\sqrt{w_2}}
\end{aligned}\right\}
$$

SOLUTION

$$w_1(e) = 4 + g(e), \quad w_2(e) = 8 + 2g(e),$$

WHERE

$$g(e) = \frac{(18\sqrt{2}+11)e^4 - (148-24\sqrt{2})e^3 - (798+36\sqrt{2})e^2 + (798-216\sqrt{2})e + 152 - 144\sqrt{2}}{(2e^2+16e-4)^2}$$

HENCE

$$E\pi(X(e)-W(e)) = p(e)\sqrt{4-w_1(e)} + (1-p(e))\sqrt{8-w_2(e)}$$

$$= p(e)\sqrt{g(e)} + (1-p(e))\sqrt{2g(e)}$$

$$= [p(e) + \sqrt{2}(1-p(e))]\sqrt{g(e)}$$

A NUMERICAL SOLUTION TO THE EQUATION

$$\frac{d}{de}\left\{\left[p(e)+\sqrt{2}\left(1-p(e)\right)\right]\sqrt{g(e)}\right\}=0$$

is

$$e^{*}=0.74, \quad w_1(e^{*})=4-2.78, \quad w_2(e^{*})=8-2(2.78).$$