CHAPTER 3: Partial derivatives and differentiation

3-1. *Find* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ *for the following functions:*

(a) $f(x,y) = x\cos x \sin y$.
(b) $f(x,y) = e^{xy^2}$.
(c) $f(x,y) = (x^2+y^2)\ln(x^2+y^2)$.

Solution:

(a) The partial derivatives of the function $f(x,y) = x(\cos x)(\sin y)$ are

$$\frac{\partial f(x,y)}{\partial x} = \cos x \sin y - x\sin x \sin y, \qquad \frac{\partial f(x,y)}{\partial y} = x\cos x \cos y$$

(b) The partial derivatives of the function $f(x,y) = e^{xy^2}$ are

$$\frac{\partial f(x,y)}{\partial x} = y^2 e^{xy^2}, \qquad \frac{\partial f(x,y)}{\partial y} = 2xye^{xy^2}$$

(c) The partial derivatives of the function $f(x,y) = (x^2+y^2)\ln(x^2+y^2)$ are

$$\frac{\partial f(x,y)}{\partial x} = 2x\ln\left(x^2+y^2\right) + 2x, \qquad \frac{\partial f(x,y)}{\partial y} = 2y\ln\left(x^2+y^2\right) + 2y$$

3-2. *Determine the marginal-products for the following production function.*

$$F(x,y,z) = 12x^{1/2}y^{1/3}z^{1/4}$$

Solution: We compute the partial derivatives with respect to each factor,

$$\begin{aligned} \frac{\partial F}{\partial x} &= 6x^{-1/2}y^{1/3}z^{1/4} \\ \frac{\partial F}{\partial y} &= 4x^{1/2}y^{-2/3}z^{1/4} \\ \frac{\partial F}{\partial z} &= 3x^{1/2}y^{1/3}z^{-3/4} \end{aligned}$$

3-3. *Find the gradient of the following functions at the given point* p

(a) $f(x,y) = \left(a^2-x^2-y^2\right)^{1/2}$ *at* $p = (a/2, a/2)$.
(b) $g(x,y) = \ln(1+xy)^{1/2}$ *at* $p = (1,1)$.
(c) $h(x,y) = e^y\cos(3x+y)$ *at* $p = (2\pi/3, 0)$.

Solution:

(a) $\nabla(a^2-x^2-y^2)^{1/2} = \left(\frac{-x}{\sqrt{(a^2-x^2-y^2)}}, \frac{-y}{\sqrt{(a^2-x^2-y^2)}}\right)$ so the gradient is

$$\left.\left(\frac{-x}{\sqrt{(a^2-x^2-y^2)}}, \frac{-y}{\sqrt{(a^2-x^2-y^2)}}\right)\right|_{x=y=a/2} = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

(b) $\nabla(\ln(1+xy)^{1/2}) = \frac{1}{2}\left(\frac{y}{1+xy}, \frac{x}{1+xy}\right)$ so the gradient is $\left(\frac{1}{4}, \frac{1}{4}\right)$.

(c)

$$\begin{aligned} &\nabla\left(e^y\cos(3x+y)\right)\big|_{x=2\pi/3, y=0} = \\ &= \left(-3e^y\sin(3x+y), e^y\cos(3x+y) - e^y\sin(3x+y)\right)\big|_{x=2\pi/3, y=0} = (0,1) \end{aligned}$$

3-4. *Consider the function*

$$f(x,y) = \begin{cases} 2\frac{x^2+y^2}{|x|+|y|}\sin(xy) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) *Find the partial derivatives of* f *at the point* $(0,0)$.
(b) *Prove that* f *is continuous on all of* $\mathbb{R}^2$. Hint: *Use (proving it) that for* $(x,y) \neq (0,0)$ *we have that*

$$0 \leq \frac{\sqrt{x^2+y^2}}{|x|+|y|} \leq 1$$

(c) *Is* f *differentiable at* $(0,0)$*?*

Solution:

(a) The partial derivative with respect to x is

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$$

since $\sin(0) = 0$. Likewise,

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$$

so,

$$\nabla f(0,0) = (0,0)$$

(b) Note that for $(x,y) \neq (0,0)$, the function f is a quotient of continuous functions and, hence is continuous for $(x,y) \neq (0,0)$.
Recall that the function f is continuous at the point $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

We prove this next. First, since $|x|, |y| \geq 0$ then

$$\begin{aligned} x^2+y^2 &\leq |x|^2+|y|^2+2|x||y| \\ &= (|x|+|y|)^2 \end{aligned}$$

Hence,

$$0 \leq \frac{\sqrt{x^2+y^2}}{|x|+|y|} \leq 1)$$

Given $\varepsilon > 0$, we take $\delta = \varepsilon/2$. If $0 < \sqrt{x^2+y^2} < \delta$ then

$$\begin{aligned} |f(x,y)| =& 2\frac{x^2+y^2}{|x|+|y|}|\sin(xy)| \qquad \text{(since } |\sin(xy)| \leq 1) \\ \leq& 2\frac{x^2+y^2}{|x|+|y|} \\ =& 2\sqrt{x^2+y^2}\frac{\sqrt{x^2+y^2}}{|x|+|y|} \qquad \text{(by the above observation)} \\ \leq& 2\sqrt{x^2+y^2}| < 2\delta = \varepsilon \end{aligned}$$

(c) First, we note that f is differentiable in $\mathbb{R}^2 \setminus \{(0,0)\}$, because the partial derivatives exist and are continuous at every point of $\mathbb{R}^2 \setminus \{(0,0)\}$. The function is differentiable at $(0,0)$ if

$$\lim_{(v_1,v_2)\to(0,0)} \frac{f(v_1,v_2) - f(0,0) - \nabla f(0,0)\cdot(v_1,v_2)}{\sqrt{v_1^2+v_2^2}} = 0$$

Note that $f(0,0) = 0$, $\nabla f(0,0)\cdot(v_1,v_2) = 0$. So, let us consider the quotient

$$\frac{f(v_1,v_2)}{\sqrt{v_1^2+v_2^2}} = 2\frac{v_1^2+v_2^2}{(|v_1|+|v_2|)\sqrt{v_1^2+v_2^2}}\sin(v_1v_2) = 2\frac{\sqrt{v_1^2+v_2^2}}{(|v_1|+|v_2|)}\sin(v_1v_2)$$

con $(v_1,v_2) \neq (0,0)$. The function f is differentiable at $(0,0)$ if

$$\lim_{(v_1,v_2)\to(0,0)} \frac{\sqrt{v_1^2+v_2^2}}{(|v_1|+|v_2|)}\sin(v_1v_2) = 0$$

By the observation made in the previous part, we have that

$$0 \leq \frac{\sqrt{v_1^2+v_2^2}}{(|v_1|+|v_2|)}|\sin(v_1v_2)| \leq |\sin(v_1v_2)|$$

and since $\sin(v_1 v_2)$ is continuous,

$$\lim_{(v_1,v_2)\to(0,0)} \sin(v_1 v_2) = 0$$

Hence,

$$\lim_{(v_1,v_2)\to(0,0)} \frac{\sqrt{v_1^2+v_2^2}}{(|v_1|+|v_2|)} \sin(v_1 v_2) = 0$$

and f is differentiable at $(0,0)$.

3-5. *Consider the function*

$$f(x,y) = \begin{cases} \frac{x\sin y}{x^2+y^2} & if (x,y) \neq (0,0) \\ 0 & if (x,y) = (0,0) \end{cases}$$

(a) *Study the continuity of f in $\mathbb{R}^2$.*
(b) *Compute the partial derivatives of f at the point $(0,0)$.*
(c) *At which points is f differentiable?*

Solution:
(a) Note that

$$\lim_{t\to 0} f(t,0) = \lim_{t\to 0} \frac{0}{t^2} = 0$$

On the other hand since,

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t\sin t}{2t^2} = \lim_{t\to 0} \frac{\sin t}{2t} = \frac{1}{2}$$

do not coincide, the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist, and hence f is not continuous at $(0,0)$. The function is continuous at $\mathbb{R}^2 \setminus \{(0,0)\}$ since it is a quotient of continuous functions and the denominator does not vanish there.
(b) The partial derivative with respect to x is

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t^3} = 0$$

since $\sin(0) = 0$. Similarly,

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t^3} = 0$$

(c) First, we note that f is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$, since the partial derivatives exist and are continuous there. The function is not differentiable at $(0,0)$ because it is not continuous at that point.

3-6. *Consider the function*

$$f(x,y) = \begin{cases} 2\frac{x^3 y}{x^2+2y^2}\cos(xy) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) *Find the partial derivatives of f at the point $(0,0)$.*
(b) *Prove that f is continuous on all of $\mathbb{R}^2$.* Hint: *Note that for $(x,y) \neq (0,0)$ we have that*

$$\frac{1}{x^2+2y^2} \leq \frac{1}{x^2+y^2}$$

(c) *Is f differentiable at $(0,0)$?*

Solution:
(a) The partial derivatives are

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t^3} = 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{2t^3} = 0$$

(b) The function is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$, since it is a quotient of continuous functions and the denominator does not vanish. Let us study the continuity at the point $(0,0)$. Let $\varepsilon > 0$. Take $\delta = \sqrt{\varepsilon/2}$. If $0 < \sqrt{x^2+y^2} < \delta$ then,

$$\begin{aligned}\left|2\frac{x^3y}{x^2+2y^2}\cos(xy)\right| &= 2\frac{x^2|x||y|}{x^2+2y^2}|\cos(xy)| \\ &\le 2|x||y| \qquad (\text{since } x^2 \le x^2+2y^2 \text{ y } |\cos(xy)| \le 1 \\ &= 2\sqrt{x^2}\sqrt{y^2} \le 2\left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+y^2}\right) < 2\delta^2 = \varepsilon\end{aligned}$$

(c) The function is differentiable at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \nabla f(0,0)\cdot(x,y)}{\sqrt{x^2+y^2}} = 0$$

Since, $f(0,0) = 0$, $\nabla f(0,0)\cdot(x,y) = 0$, the function is differentiable at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{(x^2+2y^2)\sqrt{x^2+y^2}}\cos(xy) = 0$$

Given $\varepsilon > 0$, we take $\delta = \varepsilon$. If $0 < \sqrt{x^2+y^2} < \delta$ then,

$$\begin{aligned}\left|\frac{x^3y}{(x^2+2y^2)\sqrt{x^2+y^2}}\cos(xy)\right| &\le \left|\frac{x^3y}{(x^2+2y^2)\sqrt{x^2+y^2}}\right| \\ &= \frac{x^2|x||y|}{(x^2+2y^2)\sqrt{x^2+y^2}} \\ &\le \frac{|x||y|}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2}\sqrt{y^2}}{\sqrt{x^2+y^2}} \\ &\le \sqrt{y^2} \le \sqrt{x^2+y^2} < \delta = \varepsilon\end{aligned}$$

so f is differentiable at $(0,0)$.

3-7. *Compute the derivatives of the following functions at the given point p along the vector v*
(a) $f(x,y) = x + 2xy - 3y^2$, $p = (1,2)$, $v = (3,4)$.
(b) $g(x,y) = e^{xy} + y\tan^{-1} x$, $p = (1,1)$, $v = (1,-1)$.
(c) $h(x,y) = \left(x^2+y^2\right)^{1/2}$, $p = (0,5)$, $v = (1,-1)$.

Solution:
(a) $\nabla(x+2xy-3y^2)\big|_{x=1,y=2} = (1+2y, 2x-6y)\big|_{x=1,y=2} = (5,-10)$. So, the derivative along the vector $(3,4)$ is

$$(5,-10)\cdot(3,4) = -25$$

(b) $\nabla(e^{xy} + y\arctan x)\big|_{x=1,y=1} = \left(ye^{xy} + \frac{y}{1+x^2}, xe^{xy} + \arctan x\right)\Big|_{x=1,y=1} = (e+\frac{1}{2}, e+\arctan 1) = (e+\frac{1}{2}, e+\frac{\pi}{4})$. So, the derivative along the vector $(1,-1)$ is

$$(e+\frac{1}{2}, e+\frac{\pi}{4})\cdot(1,-1) = \frac{1}{2} - \frac{\pi}{4}$$

(c) $\nabla((x^2+y^2)^{1/2})\big|_{x=0,y=5} = \left(\frac{x}{\sqrt{(x^2+y^2)}}, \frac{y}{\sqrt{(x^2+y^2)}}\right)\Big|_{x=0,y=5} = (0,1)$. So, the derivative along the vector $(1,-1)$ is

$$(0,1)\cdot(1,-1) = -1$$

3-8. *Let $B(x,y) = 10x - x^2 - \frac{1}{2}xy + 5y$ be the profits of a firm. Last year the company sold $x = 4$ units of good 1 and $y = 2$ units of good 2. This year, the company can change slightly the amounts of the goods x and y it sells. If it wishes to increase its profit as much as possible, what should $\frac{\Delta x}{\Delta y}$ be?*

Solution:

$$\nabla(10x - x^2 - \frac{xy}{2} + 5y)\Big|_{x=4,y=2} = \left(10 - 2x - \frac{y}{2}, -\frac{x}{2} + 5\right)\Big|_{x=4,y=2} = (1,3)$$

Since the gradient points in the direction of maximum growth of the function, if there is an increase $(\triangle x, \triangle y)$, for the function to increase the most, we must have that $(\triangle x, \triangle y) = k(1,3)$. From here we obtain that $\triangle x = k$ y $\triangle y = 3k$. Hence, $\triangle x/\triangle y = 1/3$.

3-9. *Knowing that* $\frac{\partial f}{\partial x}(2,3) = 7$ *and* $D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})} f(2,3) = 3\sqrt{5}$*, find* $\frac{\partial f}{\partial y}(2,3)$ *and* $D_v f(2,3)$ *with* $v = (\frac{3}{5}, \frac{4}{5})$*.*

Solution: We know that

$$D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})} f(2,3) = \left(\frac{\partial f}{\partial x}(2,3), \frac{\partial f}{\partial y}(2,3)\right) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = 3\sqrt{5}$$

and also

$$\frac{\partial f}{\partial x}(2,3) = 7$$

Letting

$$z = \frac{\partial f}{\partial y}(2,3)$$

we have that

$$(7, z) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = 3\sqrt{5}$$

But this is equivalent to

$$\frac{7\sqrt{5}}{5} + \frac{2\sqrt{5}}{5} = 3\sqrt{5}$$

And, therefore

$$\frac{\partial f}{\partial y}(2,3) = 4$$

We may compute now

$$D_{(\frac{3}{5}, \frac{4}{5})} f(2,3) = \left(\frac{\partial f}{\partial x}(2,3), \frac{\partial f}{\partial y}(2,3)\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = (7,4) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{37}{5}$$

3-10. *Find the derivative of* $f(x,y,z) = xy^2 + z^2 y$*, along the vector* $v = (1,-1,2)$ *at the point* $(1,1,0)$*. Determine the direction which maximizes (resp. minimizes) the directional derivative at the point* $(1,1,0)$*. What are the largest and smallest values of the directional derivative at that point?*

Solution: The gradient of the function $f(x,y,z) = xy^2 + z^2 y$ at the point $(1,1,0)$ is

$$\nabla f(1,1,0) = \nabla(xy^2 + z^2 y)\big|_{x=1,y=1,z=0} = \left(y^2, 2xy + z^2, 2zy\right)\big|_{x=1,y=1,z=0} = (1,2,0)$$

The derivative along v is

$$D_v f(1,1,0) = \nabla f(1,1,0) \cdot v = (1,2,0) \cdot (1,-1,2) = -1$$

The direction which maximizes the directional derivative is

$$\frac{\nabla f(1,1,0)}{\|\nabla f(1,1,0)\|} = \frac{1}{\sqrt{5}}(1,2,0)$$

and the maximum value of the directional derivative is $\|\nabla f(1,1,0)\| = \sqrt{5}$.

Likewise, the direction which minimizes the directional derivative is

$$-\frac{\nabla f(1,1,0)}{\|\nabla f(1,1,0)\|} = \frac{1}{\sqrt{5}}(-1,-2,0)$$

and the minimum value of the directional derivative is $-\|\nabla f(1,1,0)\| = -\sqrt{5}$.

3-11. *Consider the function* $f(x,y) = x^2 + y^2 + 1$ *y* $g(x,y) = (x+y, ay)$*. Determine:*

(a) *The value of* a *for which the function* $f \circ g$ *grows fastest in the direction of the vector* $v = (5,7)$ *at the point* $p = (1,1)$*.*

(b) *The equations of the tangent and normal lines to the curve* $xy^2 - 2x^2 + y + 5x = 6$ *at the point* $(4,2)$*.*

Solution: Consider the functions $f(x,y) = x^2 + y^2 + 1$ y $g(x,y) = (x+y, ay)$

(a) Their composition is $f(g(x,y)) = f(x+y, ay) = (x+y)^2 + a^2y^2 + 1$ and the gradient at the point $(1,1)$ is

$$\nabla(f(g(1,1)) = \left(2x+2y, 2x+2y+2a^2y\right)\Big|_{x=1,y=1} = (4, 4+2a^2)$$

If we want that direction of the vector $v = (5,7)$ is the direction of maximum growth of $f(g(x,y))$ at the point $(1,1)$, we must have that v and $\nabla(f(g(x,y))(1,1)$ are parallel. That is,

$$\frac{4+2a^2}{4} = \frac{7}{5}$$

whose solution is

$$a = \pm\frac{2}{\sqrt{5}}$$

(b) Note first that the point $(4,2)$ satisfies the equation $xy^2 - 2x^2 + y + 5x = 6$. Now, the gradient of the function $g(x,y) = xy^2 - 2x^2 + y + 5x = 6$ at the point $(4,2)$ is

$$\nabla g(4,2) = (y^2 - 4x + 5, 2xy + 1)\Big|_{\substack{x=4\\y=2}} = (-7, 17)$$

Thus, the equation of the tangent line is

$$(-7,17)\cdot(x-4, y-2) = 0$$

and the parametric equations of the normal line are

$$(x(t), y(t)) = (4,2) + t(-7,17)$$

3-12. *Find the Jacobian matrix of F in the following cases.*
(a) $F(x,y,z) = (xyz, x^2z)$
(b) $F(x,y) = (e^{xy}, \ln x)$
(c) $F(x,y,z) = (\sin xyz, xz)$

Solution:
(a) The Jacobian matrix of F is

$$\mathrm{D}\,F(x,y,z) = \begin{pmatrix} yz & xz & xy \\ 2xz & 0 & x^2 \end{pmatrix}$$

(b) The Jacobian matrix of

$$\mathrm{D}\,F(x,y,z) = \begin{pmatrix} ye^{xy} & xe^{xy} \\ 1/x & 0 \end{pmatrix}$$

(c) The Jacobian matrix of

$$\mathrm{D}\,F(x,y,z) = \begin{pmatrix} yz\cos xyz & xz\cos xyz & xy\cos xyz \\ z & 0 & x \end{pmatrix}$$

3-13. *Using the chain rule compute the derivatives*

$$\frac{\partial z}{\partial r} \quad \frac{\partial z}{\partial \theta}$$

in the following cases.
(a) $z = x^2 - 2xy + y^2,\ x = r + \theta,\ y = r - \theta$
(b) $z = \sqrt{25 - 5x^2 - 5y^2},\ x = r\cos\theta,\ y = r\sin\theta$

Solution:
(a)

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= 2x - 2y - 2x + 2y = 0 \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\ &= 2x - 2y - (-2x + 2y) = 4(x-y) = 8\theta \end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\
&= -\frac{10x}{2\sqrt{25-5x^2-5y^2}}\cos\theta - \frac{10y}{2\sqrt{25-5x^2-5y^2}}\sin\theta \\
&= -\frac{5r\cos^2\theta}{\sqrt{25-5r^2}} - \frac{5r\sin^2\theta}{\sqrt{25-5r^2}} = -\frac{5r}{\sqrt{25-5r^2}} \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -\frac{10x}{\sqrt{25-5x^2-5y^2}} + \frac{10y}{\sqrt{25-5x^2-5y^2}} \\
&= -\frac{10x}{2\sqrt{25-5x^2-5y^2}}(-r\sin\theta) - \frac{10y}{2\sqrt{25-5x^2-5y^2}}(r\cos\theta) \\
&= \frac{5r^2\cos\theta\sin\theta}{\sqrt{25-5r^2}} - \frac{5r^2\cos\theta\sin\theta}{\sqrt{25-5r^2}} = 0
\end{aligned}$$

3-14. *Using the capital K at time t generates an instant profit of*

$$B(t) = 5(1+t)^{1/2}K$$

Suppose that capital evolves in time according to the equation $K(t) = 120e^{t/4}$. Determine the rate of change of B.

Solution:
Since

$$\frac{dK}{dt} = 30e^{t/4}$$

we see that

$$\begin{aligned}
\frac{d}{dt}B = &= \frac{5}{2}(1+t)^{-1/2}K + 5(1+t)^{1/2}\frac{dK}{dt} \\
&= 300(1+t)^{-1/2}e^{t/4} + 150(1+t)^{1/2}e^{t/4}
\end{aligned}$$

3-15. *Verify the chain rule for the function $h = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ with $x = e^t$, $y = e^{t^2}$ and $z = e^{t^3}$.*

3-16. *Verify the chain rule for the composition $f \circ c$ in the following cases.*
(a) $f(x,y) = xy$, $c(t) = (e^t, \cos t)$.
(b) $f(x,y) = e^{xy}$, $c(t) = (3t^2, t^3)$.

Solution:
(a) The functions are $f(x,y) = xy$ and $c(t) = (x(t), y(t)) = (e^t, \cos t)$. Therefore, $f(x(t), y(t)) = f(e^t, \cos t) = e^t \cos t$ and

$$\frac{d}{dt}f(x(t), y(t)) = e^t\cos t - e^t \sin t$$

Now, we compute

$$\nabla f(c(t)) \cdot \frac{dc}{dt}$$

On the one hand,

$$\nabla f(x,y) = (y, x)$$

and

$$\frac{dc}{dt} = (e^t, -\sin t)$$

Therefore,

$$\nabla f(c(t)) \cdot \frac{dc}{dt} = y(t)e^t - x(t)\sin t$$

which coincides with the computation above.

(b) The functions are $f(x,y) = e^{xy}$ and $c(t) = (x(t), y(t)) = (3t^2, t^3)$. Therefore, $f(x(t), y(t)) = f(3t^2, t^3) = e^{3t^5}$ and

$$\frac{d}{dt} f(x(t), y(t)) = 15t^4 e^{3t^5}$$

Now, we compute

$$\nabla f(c(t)) \cdot \frac{dc}{dt}$$

On the one hand,

$$\nabla f(x,y) = (ye^{xy}, xe^{xy})$$

and

$$\frac{dc}{dt} = (6t, 3t^2)$$

Therefore,

$$\nabla f(c(t)) \cdot \frac{dc}{dt} = \left(6ye^{xy}t + 3xe^{xy}t^2\right)\Big|_{x=3t^2, y=t^3} = 15t^4 e^{3t^5}$$

3-17. *Write the chain rule $h'(x)$ in the following cases.*
(a) *$h(x) = f(x, u(x,a))$, where $a \in \mathbb{R}$ is a parameter.*
(b) *$h(x) = f(x, u(x), v(x))$.*

Solution:
(a)

$$h'(x) = \frac{\partial f\,(x, u(x,a))}{\partial x} + \frac{\partial f\,(x, u(x,a))}{\partial y}\frac{\partial u(x,a)}{\partial x}$$

(b)

$$h'(x) = \frac{\partial f(x, u(x), v(x))}{\partial x} + \frac{\partial f(x, u(x), v(x))}{\partial y} u'(x) + \frac{\partial f(x, u(x), v(x))}{\partial z} v'(x)$$

3-18. *Determine the points at which the tangent plane to the surface $z = e^{(x-1)^2+y^2}$ is horizontal. Determine the equation of the tangent plane at those points.*

Solution: Consider the function of 3 variables

$$g(x,y,z) = e^{(x-1)^2+y^2} - z$$

We are asked to compute the tangent plane to the level surface

$$A = \{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 0\}$$

at the point (x,y,z) where this tangent plane is horizontal. At that point we must have that

$$\nabla g(x,y,z) = (0,0,-1)$$

Since,

$$\nabla g(x,y,z) = \left(2(x-1)e^{(x-1)^2+y^2}, 2ye^{(x-1)^2+y^2}, -1\right)$$

we must have that $x = 1$, $y = 0$. The z coordinate is

$$z = e^{(x-1)^2+y^2}\Big|_{x=1,y=0} = 1$$

And the tangent is horizontal at the point $(1,0,1)$. The equation of the tangent plane is

$$z = 1$$

3-19. *Consider the function $f(x,y) = (xe^y)^3$.*
(a) *Compute the equation of the tangent plane to the graph of $f(x,y)$ at the point $(2,0)$.*
(b) *Using the equation of the tangent plane, find an approximation to $(1{,}999e^{0{,}002})^3$.*

Solution:

(a) We are asked to compute the tangent plane to the graph of f at the point $(2, 0, f(2, 0)) = (2, 0, 8)$. Consider the function of 3 variables

$$g(x, y, z) = x^3 e^{3y} - z$$

The graph of f is the level surface of g,

$$A = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$$

Therefore, it is enough to compute the tangent plane to the level surface A at the point $(2, 0, 8)$. Since,

$$\nabla g(2, 0, 8) = (3x^2 e^y, 3x^3 e^y, -1)\big|_{x=2,y=0,z=8} = (12, 24, -1)$$

the equation of the tangent plane is

$$(12, 24, -1) \cdot (x - 2, y, z - 8) = 0$$

that is,

$$12x + 24y - z = 16$$

(b) We are asked to estimate the function $f(x, y) = x^3 e^{3y}$ at the point $(1'999, 0'002)$. Since, f is differentiable and that point is very close to $(2, 0)$, we use Taylor's first order approximation around the point $(2, 0)$,

$$z = 12x + 24y - 16$$

and we obtain that

$$f(1'999, 0'002) \approx (12x + 24y - 16)|_{x=1'999,y=0'002} = 8'036$$

3-20. *Compute the tangent plane and normal line to the following level surfaces.*
(a) $x^2 + 2xy + 2y^2 - z = 0$ *at the point* $(1, 1, 5)$.
(b) $x^2 + y^2 - z = 0$ *at the point* $(1, 2, 5)$.
(c) $(y - x^2)(y - 2x^2) - z = 0$ *at the point* $(1, 3, 2)$.

Solution:

(a) We compute the gradient

$$\nabla(x^2 + 2xy + 2y^2 - z)\big|_{(x,y,z)=(1,1,5)} = (2x + 2y, 2x + 4y, -1)|_{(x,y,z)=(1,1,5)} = (4, 6, -1)$$

Thus, the equation of the tangent plane is

$$(4, 6, -1) \cdot (x - 1, y - 1, z - 5) = 0$$

that is,

$$4x + 6y - z = 5$$

(b) We compute the gradient

$$\nabla(x^2 + y^2 - z)\big|_{(x,y,z)=(1,2,5)} = (2x, 2y, -1)|_{(x,y,z)=(1,2,5)} = (2, 4, -1)$$

Thus, the equation of the tangent plane is

$$(2, 4, -1) \cdot (x - 1, y - 2, z - 5) = 0$$

that is,

$$2x + 4y - z = 5$$

(c) We compute the gradient

$$\begin{aligned}\nabla((y - x^2)(y - 2x^2) - z)\big|_{(x,y,z)=(1,3,2)} &= (-2x(y - 2x^2) - 4x(y - x^2), 2y - 3x^2, -1)\big|_{(x,y,z)=(1,3,2)} \\ &= (-10, 3, -1)\end{aligned}$$

Thus, the equation of the tangent plane is

$$(-10, 3, -1) \cdot (x - 1, y - 3, z - 2) = 0$$

that is,

$$10x - 3y + z = 3$$

3-21. *Compute the tangent and normal spaces to the following level surfaces.*
(a) $x^2 + 2xy + 2y^2 - z = -1,\ x^2 + 2y^2 + z = 9$ *at the point* $(-2, 0, 5)$.
(b) $x^2 - y^2 - z^2 = 2,\ x^4 + 2y^2 + z^2 = 19$ *at the point* $(2, -1, 1)$.
(c) $x^4 + xy + z^4 = 2,\ x + y^2 + 2z^2 = 1$ *at the point* $(-1, 0, 1)$.

Solution:

(a) We compute the gradients

$$\nabla(x^2+2xy+2y^2-z)\big|_{(x,y,z)=(-1,0,1)} = (2x+2y,2x+4y,-1)|_{(x,y,z)=(-1,0,1)} = (-4,-4,-1)$$

and

$$\nabla(x^2+2y^2+z)\big|_{(x,y,z)=(-1,0,1)} = (2x,4y,1)|_{(x,y,z)=(-1,0,1)} = (-4,0,1)$$

Thus, the equations of the tangent line are

$$-4(x+2)-4y-z+5=0, \quad -4(x+2)+z-5=0$$

The solution is

$$y=-2x-4, z=4x+13, \quad x\in\mathbb{R}$$

The parametric equations of the plane perpendicular to the surface at the point p are

$$\begin{aligned} x &= -2-4\lambda_1-4\lambda_2 \\ y &= -4\lambda_1 \\ z &= 5-\lambda_1+\lambda_2 \end{aligned}$$

or $x-2y+4z=18$.

(b) We compute the gradients

$$\nabla(x^2-y^2-z^2)\big|_{(x,y,z)=(-1,0,1)} = (2x,-2y,-2z)|_{(x,y,z)=(-1,0,1)} = (4,2,-2)$$

and

$$\nabla(x^4+2y^2+z^2)\big|_{(x,y,z)=(-1,0,1)} = (4x^3,4y,2z)\big|_{(x,y,z)=(-1,0,1)} = (32,-4,2)$$

Thus, the equations of the tangent line are

$$4(x-2)+2(y+1)-2(z-1)=0, \quad 32(x-2)-4(y+1)+2(z-1)=0$$

The solution is

$$y=18x-37, z=20x-39, \quad x\in\mathbb{R}$$

The parametric equations of the plane perpendicular to the surface at the point p are

$$\begin{aligned} x &= -2+4\lambda_1+32\lambda_2 \\ y &= -1+2\lambda_1-4\lambda_2 \\ z &= 1-2\lambda_1+2\lambda_2 \end{aligned}$$

or $x+18y+20z=0$.

(c) We compute the gradients

$$\nabla(x^4+xy+z^4)\big|_{(x,y,z)=(-1,0,1)} = (4x^3+y,x,4z^3)\big|_{(x,y,z)=(-1,0,1)} = (-4,-1,4)$$

and

$$\nabla(x+y^2+2z^2)\big|_{(x,y,z)=(-1,0,1)} = (1,2y,4z)|_{(x,y,z)=(-1,0,1)} = (1,0,4)$$

Thus, the equations of the tangent line are

$$-4(x+1)-y+4(z-1)=0, \quad x+4(z-1)+1=0$$

The solution is

$$y=-5x-5, z=\frac{3}{4}-\frac{x}{4}, \quad x\in\mathbb{R}$$

The parametric equations of the plane perpendicular to the surface at the point p are

$$\begin{aligned} x &= -1-4\lambda_1+\lambda_2 \\ y &= -\lambda_1 \\ z &= 1+4\lambda_1+4\lambda_2 \end{aligned}$$

or $20y-4x+z=5$.

3-22. *Let $f,g:\mathbb{R}^2\to\mathbb{R}$ be two functions with continuous partial derivatives on $\mathbb{R}^2$.*

(a) *Show that if*

$$\frac{\partial f}{\partial x}(x,y)=\frac{\partial g}{\partial x}(x,y)$$

at every point $(x,y)\in\mathbb{R}^2$, then $f-g$ depends only on y.

(b) *Show that if*

$$\frac{\partial f}{\partial y}(x,y)=\frac{\partial g}{\partial y}(x,y)$$

at every point $(x,y)\in\mathbb{R}^2$, then $f-g$ depends only on x.

(c) *Show that if* $\nabla(f-g)(x,y)=(0,0)$ *at every point* $(x,y)\in\mathbb{R}^2$, *then* $f-g$ *is constant on* $\mathbb{R}^2$.
(d) *Find a function* $f:\mathbb{R}^2\to\mathbb{R}$ *such that*

$$\frac{\partial f}{\partial y}(x,y)=yx^2+x+2y, \quad \frac{\partial f}{\partial x}(x,y)=y^2x+y, \quad f(0,0)=1$$

Are there any other functions satisfying those equations?

Solution: The functions f and g are of class C^1.
(a) Let $(a,b),(x,b)\in\mathbb{R}^2$. Let $h(x,y)=f(x,y)-g(x,y)$. By the Mean Value Theorem,

$$h(x,b)-h(a,b)=\nabla h(c)\cdot(x-a,0)$$

for some point $c=(c_1,c_2)=t(x,b)+(1-t)(a,b)=(tx+(1-t)a,b)$ with $0<t<1$. Since,

$$\frac{\partial h}{\partial x}(c)=0$$

we have that $h(x,b)-h(a,b)=0$. That is, $h(x,b)=h(a,b)$ for every $x\in\mathbb{R}$ and the function h does not depend on y
(b) Is very similar to the previous case.
(c) At each point of $\mathbb{R}^2$ we have that

$$\frac{\partial(f-g)}{\partial x}=\frac{\partial(f-g)}{\partial y}=0$$

so $f-g$ does not depend neither on x nor on y.
(d) We know that $\frac{\partial f}{\partial y}(x,y)=yx^2+x+2y$. Integrating with respect to y,

$$f(x,y)=\int(yx^2+x+2y)dy=\frac{1}{2}y^2x^2+xy+y^2+C(x)$$

where $C(x)$ is a function that depends only on x. The other condition is $\frac{\partial f}{\partial x}(x,y)=y^2x+y$. We try this with the function that we have obtained,

$$\frac{\partial}{\partial x}\left(\frac{1}{2}y^2x^2+xy+y^2+C(x)\right)=y^2x+y+C'(x)$$

so, $C'(x)=0$ and $C(x)=c$, a constant. To find c we use the condition $f(0,0)=1$. Thus,

$$\begin{aligned} f(x,y) &= \frac{1}{2}y^2x^2+xy+y^2+c \\ f(0,0) &= c \text{ and } f(0,0)=1. \text{ Hence } c=1 \end{aligned}$$

The function $f(x,y)=\frac{1}{2}y^2x^2+xy+y^2+1$ satisfies the above conditions. If there were another function g of class C^1 satisfying the same conditions, we would have that $\nabla(f-g)(x,y)=(0,0)$ at every point $(x,y)\in\mathbb{R}^2$. By part (c) there is a constant $A\in\mathbb{R}$ such that

$$(f-g)(x,y)=A$$

for every $(x,y)\in\mathbb{R}^2$. But, since $f(0,0)=1=g(0,0)$, we have that $A=0$ and the functions coincide.