Session 9
Mathematics for Economics I

Chapter 3: Differentiability. Part IV: Line and tangent planes. Taylor's polynomial of order 1

Degrees in Economics, International-Studies-and-Economics and Law-and-Economics

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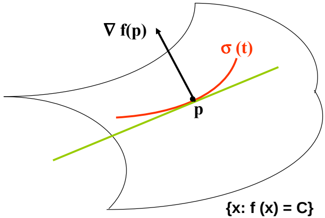
Gradient and level curves.

- Consider the level surface $S_C = \{x \in \mathbb{R}^n : f(x) = C\}$.
- Let $\sigma : \mathbb{R} \to \mathbb{R}^n$ be a differentiable curve and suppose that $\sigma(t) \in S_C$ for all $t \in \mathbb{R}$. That is $f(\sigma(t)) = c$ for every $t \in \mathbb{R}$.
- Then, $0 = \frac{d}{dt} f(\sigma(t)) = \nabla f(\sigma(t)) \cdot \frac{d\sigma}{dt}$
- That is $\nabla f(\sigma(t))$ and $d\sigma(t)/dt$ are perpendicular for every $t \in \mathbb{R}$.
- We see that $\nabla f(p)$ is perpendicular to the surface $S_C = \{x \in \mathbb{R}^n : f(x) = C\}$ at the point $p \in S_C$.



Gradient and level curves.

- This motivates the following definition: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $C \in \mathbb{R}$. Let $p \in S_C = \{x \in \mathbb{R}^n : f(x) = C\}$. Assume $\nabla f(p) \neq 0$.
- We define the tangent plane to S_C at the point p as

$$T_pS_C = \{x \in \mathbb{R}^n : \nabla f(p) \cdot (x - p) = 0$$

- And we define the line perpendicular to S_C at the point p as the line that goes through p and whose director vector is $\nabla f(p)$. That is the line whose parametric equations are

$$p + \lambda \nabla f(p), \quad \lambda \in \mathbb{R}$$

Example.

- Consider the surface given by the equation $3x^2 + 2y^2 + 5z^2 = 56$.
- The gradient of the function $f(x, y, z) = 3x^2 + 2y^2 + 5z^2$ is $\nabla f(x, y, z) = (6x, 4y, 10z)$.
- At the point $p = (-1, 2, -3)$ we get $\nabla f(-1, 2, -3) = (-6, 8, -30)$.
- The equation of the tangent plane is $-6(1 + x) + 8(-2 + y) - 30(3 + z) = 0$ or $-6x + 8y - 30z = 112$.
- The parametric equations of the normal line are $(x, y, z) = (-1, 2, -3) + t(-6, 8, -30)$.
- That is, $x = -1 - 6t$, $y = 2 + 8t$, $z = -3 - 30t$.

Gradient and level curves.

- What about a surface defined by several equations? That is, let $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ and $C_1, \ldots, C_m \in \mathbb{R}$. Let

$$S_C = \{x \in \mathbb{R}^n : f_1(x) = C_1, \ldots, f_m(x) = C_m\}$$

- Assume $\operatorname{rank}\{\nabla f_1(p), \ldots, \nabla f_m(p)\} = m$.
- Let $p \in S$. We define the 'plane' tangent to S at the point p as the intersection of the planes of the surfaces $\{x \in \mathbb{R}^n : f_1(x) = C_1\}, \ldots, \{x \in \mathbb{R}^n : f_m(x) = C_m\}$. That is,

$$T_pS = \{x \in \mathbb{R}^n : \nabla f_1(p) \cdot (x - p) = 0, \ldots, \nabla f_m(p) \cdot (x - p)\}$$

- T_pS is defined by a system of m linear equations and n unknowns. The rank of the associated matrix is m. Note that $x = p$ is a solution. So, the system is consistent.
- By the Rouchée–Frobenius Theorem, the number of parameters in the solution is $n - m$.

Gradient and level curves.

- Likewise we define the ‘plane’ perpendicular to S as the set of points of the form

$$p + \lambda_1 \nabla f_1(p), \cdots + \lambda_m \nabla f_m(p), \quad \lambda_1, \cdots, \lambda_m \in \mathbb{R}$$

- This is a subspace of dimension m.
- The $\lambda_1, \cdots, \lambda_m$ are the ‘Lagrangian multipliers’ in the courses on optimization.

Example.

- Consider the surface S determined by the equations $x^2+y^2+z^2=11, \quad x^2+2y^2-z^2=10.$
- The point $p=(3,-1,1)$ is on the surface S.
- The gradient of the functions $f_1(x,y,z)=x^2+y^2+z^2$ and $f_2(x,y,z)=x^2+2y^2-z^2$ are $\nabla f_1(x,y,z)=(2x,2y,2z)$ and $\nabla f_2(x,y,z)=(2x,4y,-2z)$.
- At the point $p=(3,-1,1)$ we get $\nabla f_1(3,-1,1)=(6,-2,2)$, $\nabla f_2(3,-1,1)=(6,-4,-2)$.
- The equation of the tangent plane to the surface S at the point p is the solution of the following linear system

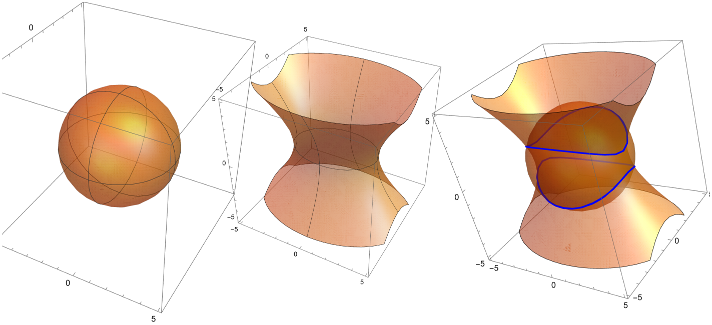
$$6(x-3)-2(y+1)+2(z-1)=0, \quad 6(x-3)-4(y+1)-2(z-1)=0$$

There is one parameter in the solution. The solution is the line

$$y=2x-7, \quad z=4-x, \quad ,z\in\mathbb{R}$$

Example.

For your amusement, here is a computer rendering of the surfaces and their intersection. You can see that the intersection is a curve, i.e. a one-dimensional object. Hence the tangent 'plane' is, in fact, a tangent line.



Example.

- The parametric equations of the normal plane are

$$(x,y,z)=(3,-1,1)+\lambda_1(6,-2,2)+\lambda_2\,(6,-4,-2)\,,\quad \lambda_1,\lambda_2\in\mathbb{R}$$

- That is,

$$\begin{aligned} x &= 3+6\lambda_1+6\lambda_2 \\ y &= -1-2\lambda_1-4\lambda_2, \quad \lambda_1,\lambda_2\in\mathbb{R} \\ z &= 1+2\lambda_1-2\lambda_2 \end{aligned}$$

- You may check that this is the plane $x+2y-z=0$.

Plane tangent to the graph of a function.

- The graph of f is the set $G = \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\}$.
- Define $g(x, y, z) = f(x, y) - z$. The graph of f may be written as $G = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$.
- An equation for the tangent plane to G at $p = (a, b)$ is

$$\nabla g(a, b, f(a, b)) \cdot ((x, y, z) - (a, b, f(a, b))) = 0$$

- Since, $\nabla g(a, b, f(a, b)) = \left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\right)$.
- We obtain

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b)$$

Taylor polynomial of first order.

- Let $f \in C^1(D)$, $p \in D$. The Taylor polynomial of first order at p is

$$P_1(x) = f(p) + \nabla f(p) \cdot (x - p)$$

- If $f(x, y)$ is a function of two variables and $p = (a, b)$, then Taylor's first order polynomial for the function f around the point $p = (a, b)$ is the polynomial

$$P_1(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b)$$

Taylor's first order polynomial.

- The function f is differentiable at (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - P_1(x,y)|}{\|(x-a, y-b)\|} = 0$$

- That is, if the tangent plane is a 'good' approximation to the value of the function

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$$

Example.

- $f(x,y) = -2y + xy^3 - 2xy + 4x - y^2 + 1$ and $p = (-1,1)$. Let us compute the equation of the tangent plane to the graph of the function f at the point $(p, f(p))$.
- The equation of the tangent plane is

$$\begin{aligned} z = f(-1,1) + \nabla f(p) \cdot (x+1, y-1) = \\ = -5 + (3,-5) \cdot (x+1, y-1) = \\ = -5 + 3(x+1) - 5(y-1) \end{aligned}$$

- It coincides with Taylor's first order polynomial of f at p.

Example.

- Consider the function $f(x, y) = 2x^2y - xy + 2x - 2y^2 - 15y + 1$ and the point $p = (1, 2)$.
- We have $\nabla f(x, y) = \left(4xy - y + 2, 2x^2 - x - 4y - 15\right)$.
- $\nabla f(1, 2) = (8, -22)$.
- Thus, the tangent plane to the graph of the function f at the point $(p, f(p))$ is

$$\begin{aligned} z &= f(1, 2) + \nabla f(p) \cdot (x - 1, y - 2) \\ &= -33 + (8, -22) \cdot (x - 1, y - 2) = \\ &= -33 + 8(-1 + x) - 22(-2 + y) \end{aligned}$$