Session 10
Mathematics for Economics II

Differentiability. Part V: Higher order derivatives. Hessian Matrix
Implicit differentiation.

Degrees in Business Administration, Finance and Accounting, Management and Technology, International Studies and Business Administration and Law and Business Administration

Universidad Carlos III de Madrid

Higher order derivatives.

- For $f : D \to \mathbb{R}$ we define the second partial derivatives as $D_{ij} f = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$.
- Likewise, we may define the higher order derivatives.
- **Example:** Let $f(x, y, z) = xy^2 + e^{zx}$.
- Then, $\frac{\partial f}{\partial x} = y^2 + ze^{zx}$, $\frac{\partial f}{\partial y} = 2xy$ and $\frac{\partial f}{\partial z} = xe^{zx}$.
- $\frac{\partial^2 f}{\partial x \partial x} = z^2 e^{zx}$, $\frac{\partial^2 f}{\partial x \partial z} = xe^{zx}$, $\frac{\partial^2 f}{\partial z \partial x} = xe^{zx}$
- Note $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}$
- This also holds for the other variables

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

Schwarz's Theorem.

Theorem

Suppose that for some $i, j = 1 \ldots, n$ *the partial derivatives*

$$\frac{\partial f}{\partial x_i}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad \frac{\partial f}{\partial x_j}, \quad \frac{\partial^2 f}{\partial x_j \partial x_i}$$

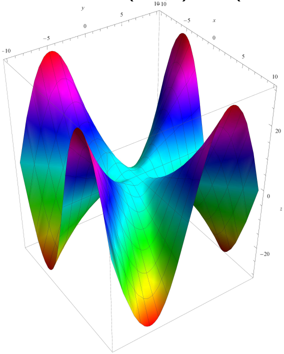
exist and are continuous in some ball $B(p, r)$*, with* $r > 0$*. Then,*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

for every x *in the ball* $B(p, r)$*.*

Schwarz's Theorem.

- Here is an example in which the assumptions of Schwarz's Theorem do not hold.
- Let $f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$



- You may check the details in the notes of Chapter IV.

Schwarz's Theorem.

- f is of class:
- $C^1(D)$ if all the first partial derivatives $\frac{\partial f}{\partial x_i}$ of f exist and are continuous on D for all $i = 1\ldots, n$.
- $C^2(D)$ if all the first partial derivatives $\frac{\partial f}{\partial x_i}$ of f exist and are of class $C^1(D)$ for every $i = 1, \ldots, n$.
- $C^k(D)$ if all the first partial derivatives $\frac{\partial f}{\partial x_i}$ of f exist and are of class $C^{k-1}(D)$ for every $i = 1, \ldots, n$.
- $C^\infty(D)$ if it is of class $C^k(D)$ for every $i, k = 1, 2, \ldots$
- All the functions we will work from now on are of class C^∞ in their domains of definition. The assumptions of Schwarz's Theorem will hold.

The Hessian matrix.

- Let $f \in C^2(D)$. The Hessian matrix of f at p is the matrix

$$\mathrm{D}^2 f(p) = \mathrm{H} f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{i,j=1,\dots,n}$$

- By Schwarz's theorem, the matrix $\mathrm{H} f(p)$ is symmetric.

The Implicit function Theorem.

- Consider the system of equations

$$f_1(u, v) = 0, \quad f_2(u, v) = 0, \quad \cdots, \quad f_m(u, v) = 0$$

- $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ will be the independent variables .
- $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ are the variables for which we want to solve for.
- We associate the following expression

$$\frac{\partial (f_1, f_2, \ldots, f_m)}{\partial (v_1, \ldots, v_m)} = \det \begin{pmatrix} \frac{\partial f_1}{\partial v_1} & \cdots & \frac{\partial f_1}{\partial v_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial v_1} & \cdots & \frac{\partial f_m}{\partial v_m} \end{pmatrix}$$

The Implicit Function Theorem I.

Theorem (The Implicit Function Theorem)

Suppose that the functions $f_1, f_2, \ldots, f_m : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *are of class* C^1 *and that there is a point* $(u_0, v_0) \in \mathbb{R}^n \times \mathbb{R}^m$ *such that*

1. $f_1(u_0, v_0) = f_2(u_0, v_0) = \cdots = f_m(u_0, v_0) = 0$*; and*
2. $\frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(v_1, \ldots, v_m)}(u_0, v_0) \neq 0.$

The Implicit Function Theorem II.

Theorem (The Implicit Function Theorem)

Then, there are an sets $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$ *and functions* $g_1, \ldots g_m : U \to \mathbb{R}$ *such that*

1. $u_0 \in U$, $v_0 \in V$.
2. *for every* $u \in U$, $f_1(u, g_1(u), \ldots, g_m(u)) = f_2(u, g_1(u), \ldots, g_m(u)) = \cdots = f_m(u, g_1(u), \ldots, g_m(u)) = 0$.
3. *If* $u \in U$ *and* $v = (v_1, \ldots, v_m) \in V$ *are solutions of the system of equations* $f_1(u, v) = f_2(u, v) = \cdots = f_m(u, v) = 0$, *then* $v_1 = g_1(u), \ldots v_m = g_m(u)$.
4. *The functions* $g_1, \ldots g_m : U \to \mathbb{R}$ *are differentiable and for each* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *we have that*

$$\frac{\partial g_i}{\partial u_j} = - \frac{\partial (f_1, f_2, \ldots, f_m)}{\partial (v_1, \ldots, v_{i-1}, u_j, v_{i+1}, \ldots, v_m)} \Bigg/ \frac{\partial (f_1, f_2, \ldots, f_m)}{\partial (v_1, \ldots, v_m)} \quad (0.1)$$

The Implicit Function Theorem III.

- Explicitly,

$$\frac{\partial (f_1, f_2, \ldots, f_m)}{\partial (v_1, \ldots, v_{i-1}, u_j, v_{i+1}, \ldots, v_m)} =$$

$$= \det \begin{pmatrix} \frac{\partial f_1}{\partial v_1} & \ldots & \frac{\partial f_1}{\partial v_{i-1}} & \frac{\partial f_1}{\partial u_j} & \frac{\partial f_1}{\partial v_{i+1}} & \ldots & \frac{\partial f_1}{\partial v_m} \\ \vdots & \ldots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial v_1} & \ldots & \frac{\partial f_m}{\partial v_{i-1}} & \frac{\partial f_m}{\partial u_j} & \frac{\partial f_1}{\partial v_{i+1}} & \ldots & \frac{\partial f_m}{\partial v_m} \end{pmatrix}$$

- The conclusion of the implicit function Theorem may be expressed in the following way,
  1. The functions $v_1 = g_1(u), v_2 = g_1(u), \ldots, v_m = g_m(u)$ are the solution of the system of equations.
  2. The derivatives of the functions $g_1, \ldots g_m : U \to \mathbb{R}$ may be computed by implicitly differentiating the system of equations and applying the chain rule.
  3. Applying several times the implicit function Theorem we may also compute the higher order derivatives of the dependent variables.

The Implicit Function Theorem. Example.

- Let

$$\begin{aligned} x^2 + ze^{xy} + z &= 1 \\ 3x + 2y + z &= 3 \end{aligned} \tag{0.2}$$

- $x = 1$, $y = z = 0$ is a solution of the system.
- Also

$$\frac{\partial (f_1, f_2)}{\partial (y, z)}(1,0,0) = \det \begin{pmatrix} xze^{xy} & e^{xy}+1 \\ 2 & 1 \end{pmatrix}\Bigg|_{x=1, y=z=0} =$$

$$= (xze^{xy} - 2e^{xy} - 2)\big|_{x=1, y=z=0} = -4 \neq 0$$

- The implicit function Theorem guarantees that we may solve for the variables y and z as functions of x for values of x near $x = 1$.

The Implicit Function Theorem. Example.

- Furthermore, differentiating with respect to x in the system we obtain

$$\begin{aligned} 2x + z' e^{xy} + z(y + xy')e^{xy} + z' &= 0 \\ 3 + 2y' + z' &= 0 \end{aligned} \tag{0.3}$$

- Now substitute $x = 1, y = z = 0$,

$$\begin{aligned} 2 + 2z'(1) &= 0 \\ 3 + 2y'(1) + z'(1) &= 0 \end{aligned} \tag{0.4}$$

- So that $z'(1) = y'(1) = -1$.

The Implicit Function Theorem. Example.

- This could be computed as well using formula 0.1,

$$y'(1) = -\frac{\frac{\partial(f_1,f_2)}{\partial(x,z)}(1,0,0)}{-4} = \frac{1}{4}\det\begin{pmatrix} 2x + yze^{xy} & e^{xy}+1 \\ 3 & 1 \end{pmatrix}\Bigg|_{x=1,y=z=0} =$$

$$= \frac{-4}{4} = -1$$

and

$$z'(1) = -\frac{\frac{\partial(f_1,f_2)}{\partial(y,x)}(1,0,0)}{-4} =$$

$$= \frac{1}{4}\det\begin{pmatrix} xze^{xy} & 2x + yze^{xy} \\ 2 & 3 \end{pmatrix}\Bigg|_{x=1,y=z=0} = \frac{-4}{4} = -1$$

The Implicit Function Theorem. Example.

- To compute the second derivatives $y''(x)$ y $z''(x)$, we differentiate each equation of the system 0.3 with respect to x.
- After simplifying we obtain

$$2+z''e^{xy}+2z'(y+xy')e^{xy}+z(2y'+xy'')e^{xy}+z(y+xy')^2e^{xy}+z''=0$$

$$2y''+z''=0$$

- Substituting $x=1, y(1)=z(1)=0$, $z'(1)=y'(1)=-1$

$$\begin{aligned} 2+2z''(1) &= 0 \\ 2y''(1)+z''(1) &= 0 \end{aligned}$$

- We obtain $z''(1)=-1$, $y''(1)=1/2$.
- Iterated differentiation allows us to obtain the derivatives of any order $z^{(n)}(1)$, $y^{(n)}(1)$.

The Implicit Function Theorem. Example.

- Consider the equation

$$x^2z^2 + 2yz + z^4 + 2 = 0$$

- We will prove that the above equation determines implicitly a differentiable function $z(x, y)$ in a neighborhood of the point $(x_0, y_0, z_0) = (-1, -2, 1)$.
- We first remark that $(x_0, y_0, z_0) = (-1, -2, 1)$ is a solution of the system of equations. The function $f(x, y, z) = x^2z^2 + 2yz + z^4 + 2$ is of class C^∞.
- We compute

$$\left|\frac{\partial f}{\partial z}\right|_{(x,y,z)=(-1,-2,1)} = \left|2x^2z + 2y + 4z^3\right|_{(x,y,z)=(-1,-2,1)} = 2$$

- By the implicit function theorem, the above system of equations determines implicitly a differentiable function $z(x, y)$ in a neighborhood of the point $(x_0, y_0, z_0) = (-1, -2, 1)$.

The Implicit Function Theorem. Example.

- We compute

$$\frac{\partial z}{\partial y}(-1,-2),$$

- Differentiating the equation implicitly with respect to y,

$$2x^2\frac{\partial z}{\partial y}z+4\frac{\partial z}{\partial y}z^3+2y\frac{\partial z}{\partial y}+2z=0$$

- We plug in the values $(x_0,y_0,z_0)=(-1,-2,1)$ to obtain the following

$$2\frac{\partial z}{\partial y}+2=0$$

- So,

$$\frac{\partial z}{\partial y}(-1,-2)=-1$$

The Implicit Function Theorem. Example.

- We compute

$$\frac{\partial z}{\partial x}(-1,-2),$$

- Differentiating the equation implicitly with respect to x,

$$-2x^2\frac{\partial z}{\partial x}z+2xz^2+4\frac{\partial z}{\partial x}z^3+2y\frac{\partial z}{\partial x}$$

- We plug in the values $(x_0,y_0,z_0)=(-1,-2,1)$ to obtain the following

$$2\frac{\partial z}{\partial x}-2=0$$

- So,

$$\frac{\partial z}{\partial x}(-1,-2)=1$$