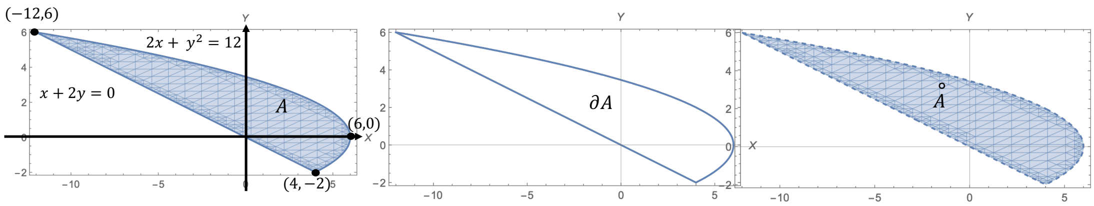
(1) Consider the set

$$A = \{(x,y) \in \mathbb{R}^2 : 2x + y^2 - 12 \leq 0, x + 2y \geq 0\}$$

and the function

$$f(x,y) = e^{\frac{x}{2}+y}$$

(a) **(20 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded and/or compact.
Solution: *The set A, its interior and its boundary are approximately as indicated in the picture.*



And the closure coincides with A, $\bar{A} = A$. The functions $h_1(x,y) = 2x+y^2-12$ and $h_2(x,y) = x+2y$ are continuous and $A = \{(x,y) \in \mathbb{R}^2 : h_1(x,y) \leq 0, h_2(x,y) \leq 0\}$. Hence, the set A is closed (Note also that $\partial A \subset A$). It is not open because $A \cap \partial A \neq \emptyset$. The set A is bounded. Therefore, the set A is compact.

(b) **(10 points)** Prove that the set A is convex.
Solution: *The set A is the intersection of two sets $A = A_1 \cap A_2$ with*

$$A_1 = \{(x,y) \in \mathbb{R}^2 : h_1(x,y) \leq 0\}$$

$$A_2 = \{(x,y) \in \mathbb{R}^2 : h_2(x,y) \geq 0\},$$

where the functions functions $h_1(x,y) = 2x + y^2 - 12$ and $h_2(x,y) = x + 2y$ have been defined in the previous part. The function h_2 is linear. Hence, the set A_2 is convex. We show next that the set A_1 is also convex. Consider the function $g(x,y) = 2x + y^2 - 12$. Its Hessian matrix is

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

which is positive semidefinite in all of $\mathbb{R}^2$. Hence, the function g is convex and the set A_1 is convex. Since, A is the intersection of convex sets, it is also convex.

(c) **(5 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.
Solution: *The set A is compact and the function $f(x,y) = e^{\frac{x}{2}+y}$ is continuous. Hence, Weierstrass Theorem applies.*

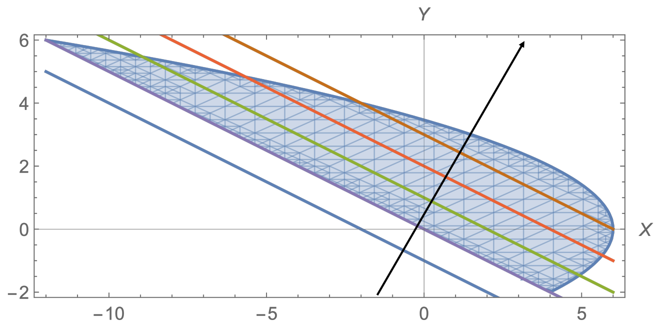
(d) **(10 points)** Draw the level curves of f, indicating the direction of growth of the function.
Solution: *For $D > 0$, the level curves*

$$f(x,y) = e^{\frac{x}{2}+y} = D$$

are straight lines of the form

$$y = \frac{\ln D}{2} - \frac{x}{2} = C - \frac{x}{2}$$

Graphically,

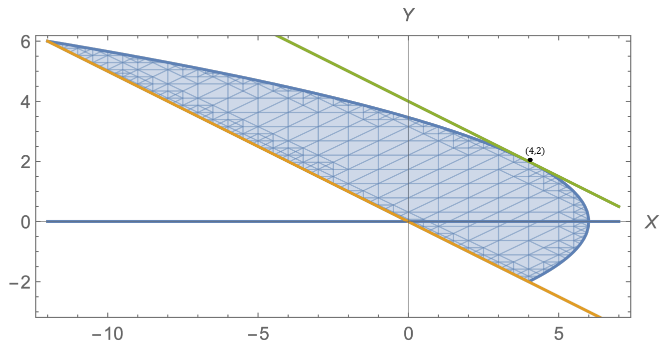


The black arrow represents the direction of growth of the function f.

(e) **(20 points)** Using the level curves of f, determine (if they exist) the points of the set A where the function f attains its **global minimum** and/or a **global maximum** values on the set A.
Solution: *Graphically, the maximum value is attained at the point (x_0, y_0) where the line $y = C - \frac{x}{2}$ is tangent to the graph of the curve $2x + y^2 = 12$. Let $y(x)$ be the function defined by the equation $2x + y^2 = 12$ near the point (x_0, y_0). The slope of the line $y = C - \frac{x}{2}$ is $m = -\frac{1}{2}$. Hence, $y'(x_0,) = -\frac{1}{2}$. Taking derivatives, implicitly, we have*

$$0 = 2 + 2y_0 y'(x_0) = 2 - y_0$$

Thus, $y_0 = 2$. Substituting this value in the equation $2x_0 + y_0^2 = 12$, we obtain $x_0 = 4$. The maximum value is $f(4, 2) = e^4$.



Graphically, the minimum value is attained at the line $y = -\frac{1}{2}x$. And the maximum value is $f\left(x, -\frac{1}{2}x\right) = e^0 = 1$.

(2) Consider the function $f(x,y,z) = -x^2 - 2xy - y^3 + 4yz - z^3$ defined in $\mathbb{R}^3$.

(a) **(15 points)** Determine D, the largest open set of $\mathbb{R}^3$ where the function f is strictly concave.

Solution: *We have*

$$\nabla f(x,y,z) = \left(-2x - 2y, -2x - 3y^2 + 4z, 4y - 3z^2\right),$$

$$\mathrm{H}(f)(x,y,z) = \begin{pmatrix} -2 & -2 & 0 \\ -2 & -6y & 4 \\ 0 & 4 & -6z \end{pmatrix}$$

We obtain $D_1 = -2$, $D_2 = 12y - 4$, $D_3 = 32 + 24z - 72yz$. *We require that*

- $D_1 < 0$. *It holds in* $\mathbb{R}^3$.
- $D_2 > 0$. *Hence,* $y > \frac{1}{3}$.
- $D_3 < 0$. *Hence,* $z > \frac{4}{3(3y-1)}$.

Therefore,

$$D = \left\{(x,y,z) \in \mathbb{R}^3 : \quad y > \frac{1}{3}, z > \frac{4}{3(3y-1)}\right\}$$

(b) **(10 points)** Prove that the set D found in the previous part is convex.

Solution: *Let* $D_1 = \left\{(x,y,z) \in \mathbb{R}^3 : y > \frac{1}{3}, \right\}$, $D_2 = \left\{(x,y,z) \in \mathbb{R}^3 : z > \frac{4}{3(3y-1)}\right\}$. *We see that* D_1 *is convex. Consider the function* $g(y) = \frac{4}{3(3y-1)}$. *We have* $g''(x) = \frac{24}{(3y-1)^3} > 0$ *for* $y > \frac{1}{3}$. *Therefore,* $g(x)$ *is a convex function and* D_2 *is convex. We conclude that* $D = D_1 \cap D_2$ *is also convex.*

(3) Consider the system of equations

$$\begin{aligned} xy + 2xz^2 + 6 &= 0 \\ x^2 + y + 2z &= 7 \end{aligned}$$

(a) **(10 points)** Using the implicit function theorem, prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (-1, 4, 1)$.

Solution: *We first remark that* $(x_0, y_0, z_0) = (-1, 4, 1)$ *is a solution of the system of equations. The functions* $f_1(z,t) = ty - xy + xz - 3$ *and* $f_2(z,t) = t^2x + z - 2$ *are of class* C^∞*, because they are polynomials. We compute*

$$\left|\frac{\partial(f_1,f_2)}{\partial(y,z)}\right|_{(z,t)=(-1,4,1)} = \begin{vmatrix} x & 4xz \\ 1 & 2 \end{vmatrix}_{(z,t)=(-1,4,1)} = \begin{vmatrix} -1 & -4 \\ 1 & 2 \end{vmatrix} = 2$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ *and* $z(x)$ *in a neighborhood of the point* $(x_0, y_0, z_0) = (-1, 4, 1)$.

(b) **(15 points)** Compute $y'(-1)$, $z'(-1)$.
Solution: *Differentiating implicitly the system with respect to* x,

$$\begin{aligned} 0 &= xy'(x) + y(x) + 4xz(x)z'(x) + 2z(x)^2 \\ 0 &= y'(x) + 2z'(x) + 2x \end{aligned}$$

We plug in the values $(x_0, y_0, z_0) = (-1, 4, 1)$ *to obtain*

$$\begin{aligned} 0 &= -y'(-1) - 4z'(-1) + 6 \\ 0 &= y'(-1) + 2z'(-1) - 2 \end{aligned}$$

Therefore

$$y'(-1) = -2, \quad z'(-1) = 2$$

(c) **(10 points)** Compute Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = -1$.
Solution: *Taylor's polynomial of order 1 of the function* $y(x)$ *at the point* $x_0 = -1$ *is*

$$P_1(x) = y(x_0) + y'(x_0)(x - x_0) = 4 - 2(1 + x)$$

Taylor's polynomial of order 1 of the function $z(x)$ *at the point* $x_0 = -1$ *is*

$$P_2(x) = z(x_0) + z'(x_0)(x - x_0) = 1 + 2(1 + x)$$

(d) **(5 points)** Use Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = -1$ to give an approximate value of $y(-0.99)$ and $z(-0.99)$.
Solution: *We have*

$$P_1(-0.99) = 4 - 2 \times 0.01 = 3.98$$

and

$$P_2(-0.99) = 1 + 2 \times 0.01 = 1.02$$

(4) Consider the quadratic form

$$Q(x, y, z) = bx^2 + 4xy - 2y^2 + 2xz + 2yz + az^2$$

where $a, b \in \mathbb{R}$.

(a) **(5 points)** Compute the symmetric matrix A associated to Q.

Solution: *The associated matrix is*

$$A = \begin{pmatrix} b & 2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & a \end{pmatrix}$$

(b) **(10 points)** Compute the leading principal minors D_1, D_2 and D_3 of A.

Solution: *We have*

$$D_1 = b, \quad D_2 = \begin{vmatrix} b & 2 \\ 2 & -2 \end{vmatrix} = -4 - 2b, \quad D_3 = |A| = 6 - 4a - b - 2ab$$

(c) **(15 points)** For what values of $a \neq 0$ and $b \neq 0$ is the quadratic form Q positive definite?

Solution: *We need $D_1 > 0, D_2 > 0, D_3 > 0$. In particular we need $b > 0$, $b < -2$ which is impossible. Hence. the quadratic form Q is no positive definite for any value of a and b.*

(d) **(15 points)** For what values of $a \neq 0$ and $b \neq 0$ is the quadratic form Q negative definite?

Solution: *We need $D_1 < 0, D_2 > 0, D_3 < 0$. That is, $b < 0$, $b < -2$ and $6 - b < 2a(b + 2)$. Since, $b + 2 < 0$, the last inequality is equivalent to $a < \frac{6-b}{2a(b+2)}$. Therefore, the quadratic form Q is negative definite for $b < -2$ and $a < \frac{6-b}{2a(b+2)}$.*

(e) **(15 points)** Classify the quadratic form Q for the values of $b = 0$, $a \in \mathbb{R}$.

Solution: *For $b = 0$, we obtain that*

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & a \end{pmatrix}$$

We consider the chain

$$D_1 = -2 < 0, \quad D_2 = \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} = -4, \quad D_3 = |A| = 6 - 4a$$

Hence, the quadratic form is indefinite.

(5) (a) **(20 points)** Consider the function $f(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the functions $x(t,z), y(t,z) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = xy + e^y \qquad \text{and} \qquad x(t,z) = t + z^2, \quad y(t,z) = tz$$

And consider the composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(t,z) = f(x(t,z), y(t,z))$. Use the chain rule to compute

$$\frac{\partial h}{\partial t}(0,1), \quad \frac{\partial h}{\partial z}(0,1)$$

Solution:

$$x(0,1) = 1, \quad y(0,1) = 0$$

$$Df(x,y) = (y, x + e^y), \quad Df(1,0) = (0,2)$$

Let $g(t,z) = (x(t,z), y(t,z)) = (t + z^2, tz)$. *We have*

$$Dg(t,z) = \begin{pmatrix} 1 & 2z \\ z & t \end{pmatrix}, \quad Dg(0,1) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

By the chain rule,

$$\begin{pmatrix} \frac{\partial h}{\partial t}(0,1) & \frac{\partial h}{\partial z}(0,1) \end{pmatrix} = Df(1,0)\,Dg(0,1) = (0,2)\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = (2,0)$$

Therefore,

$$\frac{\partial h}{\partial t}(0,1) = 2, \quad \frac{\partial h}{\partial z}(0,1) = 0$$

(b) **(10 points)** Consider two function $g(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\sigma(t) : \mathbb{R} \longrightarrow \mathbb{R}^2$. Let $h(t) = g(\sigma(t))$. It is known that $\sigma(0) = (1,-1)$, $\sigma'(0) = (4,0)$ and $h'(0) = 8$. What is the value of $\frac{\partial g}{\partial x}(1,-1)$?
Solution: *By the chain rule we have that*

$$8 = h'(0) = \nabla g(1,-1) \cdot (4,0) = \left(\frac{\partial g}{\partial x}(1,-1), \frac{\partial g}{\partial y}(1,-1)\right) \cdot (4,0) = 4\frac{\partial g}{\partial x}(1,-1)$$

Therefore,

$$\frac{\partial g}{\partial x}(1,-1) = 2$$

(6) Consider the point $p = (0, 1, -1)$ on the curve S defined by the equations

$$x^2 - 2xy + 2y^2 + z = 1 \quad x^3 - y - z^2 = -2$$

(a) **(10 points)** Compute the tangent line to the curve S at the point $p = (0, 1, -1)$.

Solution: *Let* $f(x, y, z) = x^2 - 2xy + 2y^2 + z - 1$, $g(x, y, z) = x^3 - y - z^2 + 2$. *The gradient of the functions* f *and* g *are*

$$\nabla f(x, y, z) = (2x - 2y, -2x + 4y, 1)\,, \quad \nabla f(0, 1, -1) = (-2, 4, 1)$$

and

$$\nabla g(x, y, z) = \left(3x^2, -1, -2z\right)\,, \quad \nabla g(0, 1, -1) = (0, -1, 2)$$

Hence the tangent line is the intersection of the following planes.

$$-2x + 4(y - 1) + z + 1 = 0, \quad -(y - 1) + 2(z + 1) = 0$$

(b) **(10 points)** Compute the parametric equations of the plane perpendicular to the curve S at the point $p = (0, 1, -1)$. (These equation are of the form $(x, y, z) = (x_0, y_0, z_0) + \lambda_1(a, b, c) + \lambda_2(d, e, f)$)

Solution: *The equations are*

$$(x, y, z) = (0, 1, -1) + \lambda_1(-2, 4, 1) + \lambda_2(0, -1, 2)$$

or

$$\begin{aligned} x &= -2\lambda_1 \\ y &= 1 + 4\lambda_1 - \lambda_2 \\ z &= -1 + \lambda_1 + 2\lambda_2 \end{aligned}$$

(c) **(5 points)** Compute the general (o implicit) equation of the plane perpendicular to the curve S at the point $p = (0, 1, -1)$. This equation is of the form $ax + by + cz = d$.

Solution: *We start from the equations*

$$\begin{aligned} x &= -2\lambda_1 \\ y &= 1 + 4\lambda_1 - \lambda_2 \\ z &= -1 + \lambda_1 + 2\lambda_2 \end{aligned}$$

We rewrite them as

$$\begin{aligned} 2x &= -4\lambda_1 \\ y &= 1 + 4\lambda_1 - \lambda_2 \\ 4z &= -4 + 4\lambda_1 + 8\lambda_2 \end{aligned}$$

Adding the first equation to the second and third equations, we obtain

$$\begin{aligned} 2x &= -4\lambda_1 \\ 2x + y &= 1 - \lambda_2 \\ 2x + 4z &= -4 + 8\lambda_2 \end{aligned}$$

We multiply the second equation by 8.

$$\begin{aligned} 2x &= -4\lambda_1 \\ 16x + 8y &= 8 - 8\lambda_2 \\ 2x + 4z &= -4 + 8\lambda_2 \end{aligned}$$

and add the second equation to the third one.

$$\begin{aligned} 2x &= -4\lambda_1 \\ 16x + 8y &= 8 - 8\lambda_2 \\ 18x + 8y + 4z &= 4 \end{aligned}$$

The equation of the normal plane is $18x + 8y + 4z = 4$.