University Carlos III
Mathematics for Economics I.
Final Exam. January 17th 2025

Last Name:	**Name:**
Degree:	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

(1) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 5, -3x \leq y \leq 3x\}$$

and the function

$$f(x, y) = (2y - x)^3$$

(a) **(20 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded and/or compact.
(b) **(10 points)** Prove that the set A is convex.
(c) **(5 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.
(d) **(10 points)** Draw the level curves of f, indicating the direction of growth of the function.
(e) **(20 points)** Using the level curves of f, determine (if they exist) the points of the set A where the function f attains its **global minimum** and/or a **global maximum** values on the set A.

(2) Consider the function $f(x, y, z) = -x^3 - 2xy - y^3 + 2yz - z^2$ defined in $\mathbb{R}^3$.
(a) **(15 points)** Determine D, the largest open set of $\mathbb{R}^3$ where the function f is strictly concave.

(b) **(10 points)** Prove that the set D found in the previous part is convex.

(3) Consider the system of equations

$$\begin{aligned} ty - xy + xz &= 3 \\ t^2 x + z &= 2 \end{aligned}$$

(a) **(10 points)** Using the implicit function theorem, prove that the above system of equations determines implicitly two differentiable functions $y(x, t)$ and $z(x, t)$ in a neighborhood of the point $(x_0, y_0, z_0, t_0) = (0, 3, 2, 1)$.

(b) **(15 points)** Compute

$$\frac{\partial y}{\partial x}(0, 1), \frac{\partial z}{\partial x}(0, 1), \frac{\partial y}{\partial t}(0, 1), \frac{\partial z}{\partial t}(0, 1)$$

(c) **(10 points)** Compute Taylor's polynomial of order 1 of the functions $y(x, t)$ and $z(x, t)$ at the point $(x_0, t_0) = (0, 1)$.
(d) **(5 points)** Use Taylor's polynomial of order 1 of the functions $y(x, t)$ and $z(x, t)$ at the point $(x_0, t_0) = (0, 1)$ to give an approximate value of $y(0.01, 0.99)$ and $z(0.01, 0.99)$.

(4) Consider the quadratic form

$$Q(x, y, z) = ax^2 + by^2 + 4xy + 2xz + 2yz + z^2$$

where $a, b \in \mathbb{R}$.
(a) **(5 points)** Compute the symmetric matrix A associated to Q.

(b) **(10 points)** Compute the leading principal minors D_1, D_2 and D_3 of A.

(c) **(15 points)** For what values of $a \neq 0$ and $b \neq 0$ is the quadratic form Q positive definite?

(d) **(15 points)** For what values of $a \neq 0$ and $b \neq 0$ is the quadratic form Q negative definite?

(e) **(15 points)** Classify the quadratic form Q for the values of $a = 0$, $b \in \mathbb{R}$.

(5) **(20 points)** Consider the function $f(x, y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the functions $x(t, z), y(t, z) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = xy + y^2 \qquad \text{and} \qquad x(t, z) = t^2 + z, \quad y(t, z) = tz^2$$

And consider the composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(t, z) = f(x(t, z), y(t, z))$. Use the chain rule to compute

$$\frac{\partial h}{\partial t}(-1, 1), \quad \frac{\partial h}{\partial z}(-1, 1)$$

(6) Consider the function

$$f(x, y) = 5x^4 + 3x^2 + 4xy + y^6 + 5y^2$$

(a) **(5 points)** Write the gradient and the Hessian matrix of the function f at a point (x, y).

(b) **(10 points)** Write the polynomial of Taylor of orden 1 of the function f at the point $p = (0, 1)$.

(c) **(10 points)** Write the polynomial of Taylor of orden 2 of the function f at the point $p = (0, 1)$.

(d) **(10 points)** Prove that for any $(x, y) \in \mathbb{R}^2$ the following inequality holds

$$5x^4 + 3x^2 + 4xy + y^6 + 5y^2 \geq 4x + 16y - 10$$

and the inequality is strict except for $x = 0$, $y = 1$.