Exercise	1	2	3	4	Total
Points					

Universidad Carlos III de Madrid

Department of Economics **Introduction to Mathematics I Final Exam** **January 8th 2026**

Exam time: 1 hour 35 minutes.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Let** $f(x) = \ln(e^x + e^{-x})$ **be the function, defined on the real line. Find the following:**

(a) Find the intervals of increase and decrease, the global extrema, the asymptotes, and the range of $f(x)$.

(b) Find the intervals of concavity, convexity, and the inflection points of $f(x)$. Sketch the graph of $f(x)$.

(c) Consider $f_1(x)$ and $f_2(x)$ as the function $f(x)$ restricted to the intervals where $f(x)$ is decreasing, and increasing, respectively. Sketch $f_1^{-1}(x)$ and $f_2^{-1}(x)$.

Hint for a): $\ln a - b = \ln(a/e^b)$.

0.5 points for part a); 0.3 points for part b); 0.2 points for part c)

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a) First, $f\ (x)$ is differentiable on the entire real line. Furthermore, since

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

the only critical point is $x = 0$. $f'(x) < 0$ if $x \in (-\infty, 0)$, so $f\ (x)$ is decreasing on $(-\infty, 0]$. $f'(x) > 0$ if $x \in (0, \infty)$, so $f\ (x)$ is increasing on $[0, \infty)$. Obviously, it follows that $x = 0$ is the global minimizer of $f(x)$ and $f\ (x)$ has no global maximum, as there are no other critical points.

Regarding the asymptotes, since the function is even, it is sufficient to study the problem at ∞ and apply symmetry at $-\infty$.

Step 1:

$$\lim_{x\longrightarrow\infty} f(x)/x = \lim_{x\longrightarrow\infty} \frac{\ln(e^x + e^{-x})}{x} = \frac{\infty}{\infty} \overset{\text{(L'Hôpital)}}{=} \lim_{x\longrightarrow\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\longrightarrow\infty} \frac{e^x}{e^x} \cdot \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1;$$

Step 2:

$$\lim_{x\longrightarrow\infty} (f(x) - x) = \lim_{x\longrightarrow\infty} (\ln(e^x + e^{-x}) - \ln e^x) = \lim_{x\longrightarrow\infty} \ln(\frac{e^x + e^{-x}}{e^x}) = \lim_{x\longrightarrow\infty} \ln(1 + e^{-2x}) = 0.$$

Thus, $y = x$ is the oblique asymptote of $f(x)$ at ∞ and, by symmetry, $y = -x$ is the oblique asymptote of $f(x)$ at $-\infty$.

Finally, since $f(0) = \ln 2$, $f(x)$ is increasing on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = \infty$, the range of $f(x)$ is $[\ln 2, \infty)$.

b) Since

$$f''(x) = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4e^x e^{-x}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} > 0,$$

it follows that $f(x)$ is convex on all $\mathbb{R}$ and, therefore, there are no inflection points.

Hence, the graph of the function would be, approximately, as observed in Figure 1.

c) As we have seen, the domain of $f_1(x)$ is $(-\infty, 0]$ and its range is $[\ln 2, \infty)$. On the other hand, the domain of $f_2(x)$ is $[0, \infty)$ and its range is $[\ln 2, \infty)$. Therefore, the domain of $f_1^{-1}(x)$ is $[\ln 2, \infty)$ and its range is $(-\infty, 0]$.

Thus, by symmetry with respect to the main diagonal ($y = x$), $f_1^{-1}(x)$ is decreasing and has an oblique asymptote $y = -x$, approaching it from above.

Similarly, by symmetry with respect to the main diagonal, $f_2^{-1}(x)$ is increasing and has an oblique asymptote $y = x$, approaching it from below.

Conclusion: the graphs of the inverse functions will be approximately as observed in Figure 2.

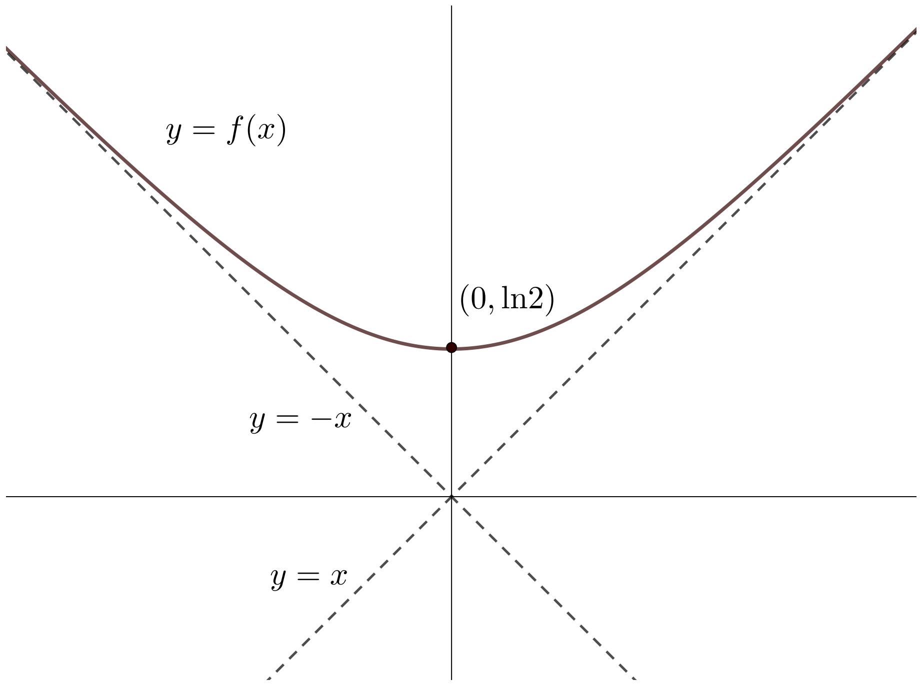

FIGURE 1. Graph of the function.

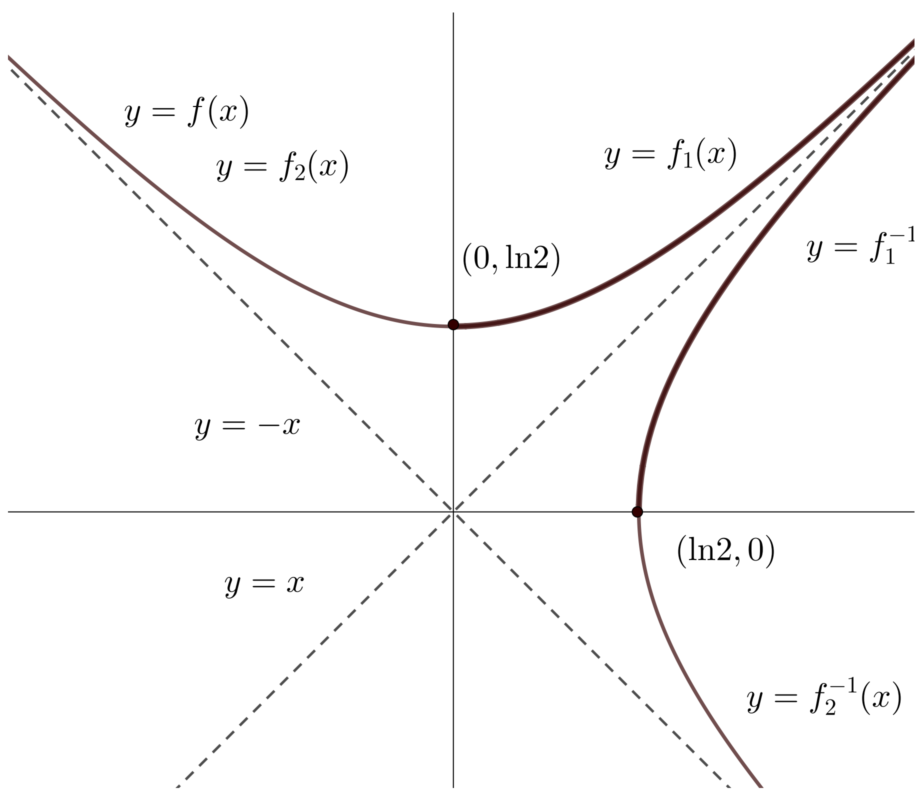

FIGURE 2. Graph of the inverse functions.

(2) **Suppose the equation $e^{y-2}+y-4x=-1$, implicitly defines the function $y=f(x)$ in a neighborhood of the point $x=1, y=2$. Find the following:**

(a) Find the tangent line and the Taylor polynomial of order 2 of f at $a=1$.

(b) Sketch the approximate graph of f near the point $x=1$.

(c) Let $\delta>0$ be sufficiently small. Calculate the approximate values of $f(1-\delta)$ and $f(1+\delta)$ using the tangent line and the Taylor polynomial of order 2. Compare these approximations with the exact values of $f(1-\delta)$ and $f(1+\delta)$.

0.4 points for part a); 0.3 points for part b); 0.3 points for part c).

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a) To find the tangent line, we differentiate the equation $F^{\sim}(x)=F(x,y=f(x))=-1$:

$$y'e^{y-2}+y'-4=y'(e^{y-2}+1)-4=0$$

Substituting $x=1$, $y(1)=2$ yields $y'(1)=f'(1)=2$. Thus, the equation of the tangent line is: $y=P_1(x)=2+2(x-1)=2x$.

Similarly, we calculate the second derivative of the equation $F^{\sim}(x)=-1$:

$$y''(e^{y-2}+1)+(y')^2e^{y-2}=0$$

Substituting $x=1, y(1)=2, y'(1)=2$ yields $y''(1)=f''(1)=-2$. Thus, the Taylor polynomial of order 2 is: $P_2(x)=2x-(x-1)^2$.

b) Using the tangent line, the Taylor polynomial of the function found in a) and observing that the function is locally concave at the point, $f''(1)<0$, the graph of the function near the point $(1,2)$ will be approximately as shown in Figure 3:

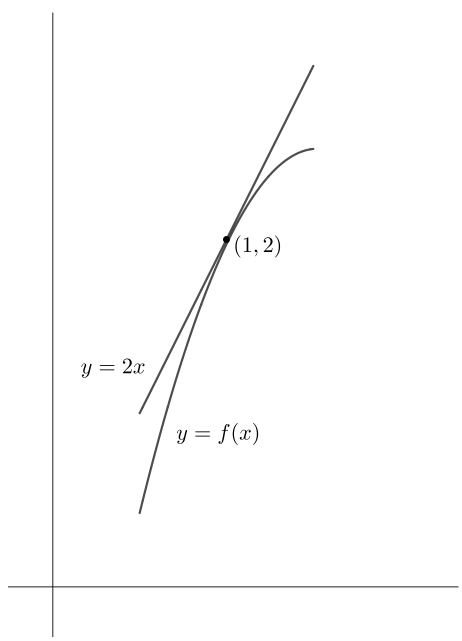

FIGURE 3. Local representation of the function at the point (1,2).

c) Using the tangent line, the approximate values of $f(1-\delta)$ and $f(1+\delta)$ will be $2-2\delta$ and $2+2\delta$, respectively. Due to the concavity of $f(x)$ in $(1-\delta,1+\delta)$, these values will be **GREATER** than the exact value of the function at those points.

On the other hand, using the Taylor polynomial of order 2, the approximate values of $f(1-\delta)$ and $f(1+\delta)$ will be $2-2\delta-\delta^2$ and $2+2\delta-\delta^2$, respectively.

These values will be a better approximation to the exact value of the function at these points, but we cannot determine if they are greater or less than the exact value without calculating a higher-order derivative of the implicit function.

(3) **Let $C(x) = 9 + 2x + x^2$ be the cost function and $p(x) = 14 - 2x$ the inverse demand function of a monopoly firm. Find the following:**

(a) Find the level of production that maximizes profit.

(b) Find the level of production that minimizes average cost.

(c) The government wants to increase the production found in a), so it proposes that the firm produces at the level found in b). What will be the reaction of the firm?

(d) The government is unsure about the firm's reaction in case c), so it proposes to cover all costs. Will production increase this way?

0.3 points for part a); 0.3 points for part b); 0.2 points for part c); 0.2 points for part d).

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a) First, we calculate the profit function:

$$B(x) = (14 - 2x)x - (9 + 2x + x^2) = -3x^2 + 12x - 9$$

If we calculate the first and second derivatives of B:

$$B'(x) = -6x + 12; \quad B''(x) = -6 < 0$$

Thus, we see that B has a single critical point at $x^* = 2$ and, since B is a concave function, this critical point is the unique global maximizer.

b) Since the average cost function is $\dfrac{C(x)}{x} = \dfrac{9}{x} + 2 + x$, its derivative is:

$$\left(\frac{C(x)}{x}\right)' = -\frac{9}{x^2} + 1 = 0 \Longleftrightarrow \hat{x} = 3,$$

which is the only valid critical point. Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{18}{x^3} > 0$, the function is convex and the critical point is the global minimizer.

c) $B(3) = 0$. Thus, the firm will be indifferent between producing or not producing.

d) In this case, $B(x) = (14 - 2x)x$, i.e., the revenue function from a). If we calculate the first and second derivatives of B:

$$B'(x) = 14 - 4x; \quad B''(x) = -4 < 0$$

Thus, we see that B has a single critical point at $x^+ = 3.5$ and, since B is a concave function, this critical point is the unique global maximizer.

Therefore, the firm will increase its production.

(4) **Let** $h(x) = xe^{2x-1} + 8(x-1)^3$. **Find the following:**

(a) State Bolzano's Zero Theorem for $h(x)$ on $[a, b]$.

(b) Study how many zeros the function $h(x)$ has on the entire real line.

Hint for b): sketch the functions $f(x) = xe^{2x-1}$ and $g(x) = -8(x-1)^3$ and consider separately the cases $x < 0$ and $x > 0$

(c) Find a zero of the function $h(x)$ with an error less than $1/4$.

0.2 points for part a); 0.5 points for part b); 0.3 points for part c).

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a) The hypotheses, or initial conditions, are that h is continuous on $[a, b]$ and that $h(a) \cdot h(b) < 0$.
The thesis, or conclusion, is that there exists $c \in (a, b)$ such that $h(c) = 0$.

b) Studying the zeros of $h(x) = f(x) - g(x) = 0 \Longleftrightarrow f(x) = g(x)$ is equivalent to finding how many times the graphs of the functions $f(x)$ and $g(x)$ intersect. $f(x) \leq 0$ if $x \leq 0$, while $g(x) \geqslant 8$ if $x \leq 0$. Thus $h(x) = f(x) - g(x) < 0$ if $x \leq 0$. (No zeros for $x \leq 0$).
On the other hand, $f(x)$ and $g(x)$ are continuous functions on the reals, $f(0) = 0 < g(0) = 8$. $f(x)$ is increasing on $[0, \infty)$, and $g(x)$ is decreasing in that same interval (in fact, $g(x)$ is decreasing on the entire real line).
Furthermore, $\lim_{x \longrightarrow \infty} f(x) = \infty$, $\lim_{x \longrightarrow \infty} g(x) = -\infty$.
Therefore, the graphs of $f(x)$ and $g(x)$ have a unique intersection point at x^*. This intersection point is the zero of the function $h(x)$.

c) Observing the sketch and testing values: $h(0) = f(0) - g(0) = -8 < 0$, $h(1) = f(1) - g(1) = e > 0$, so the zero of $h(x)$ is in the interval $(0, 1)$. Furthermore, $h(\frac{1}{2}) = f(\frac{1}{2}) - g(\frac{1}{2}) = -\frac{1}{2} < 0$, $h(1) > 0$ so the zero of $h(x)$ is in the interval $(\frac{1}{2}, 1)$.
Therefore, we can take the value $\frac{3}{4}$ as the zero of $h(x)$ with an error less than $\frac{1}{4}$.
Figure 4 clarifies the situation.

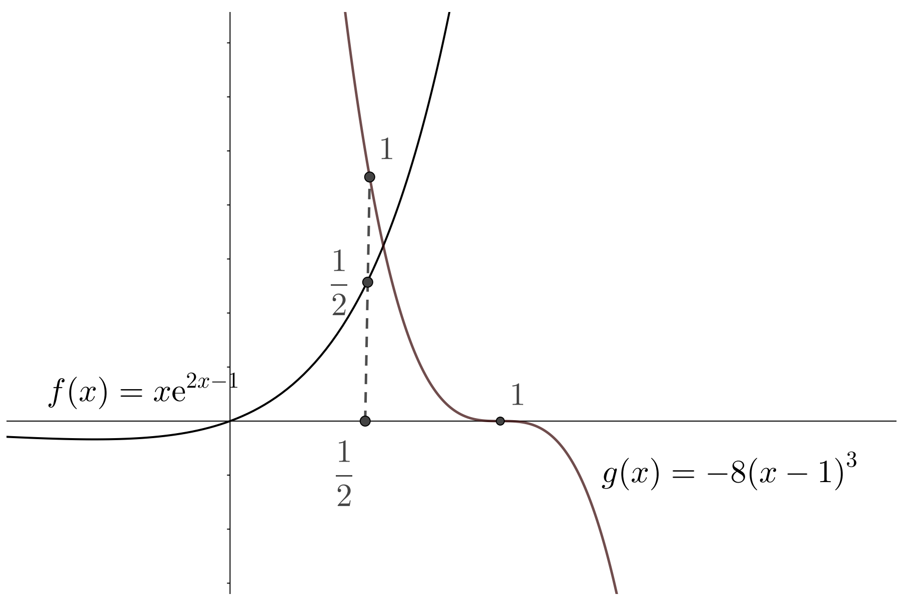

FIGURE 4. Graph of the functions $f(x)$ and $g(x)$.