Universidad Carlos III de Madrid

| **Exercise** | 1 | 2 | 3 | 4 | **Total** |
|---|---|---|---|---|---|
| **Points** | | | | | |

**Department of Economics** **Introduction to Mathematics I Extraordinary Exam** **June 16th 2026**

**Exam time: 1 hour 35 minutes**.

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| **LAST NAME:** | | **FIRST NAME:** |
| **ID:** | **DEGREE:** | **GROUP:** |

**Important note:** The use of calculators or any other electronic device is strictly prohibited. For a result to be considered in the grading, it is an indispensable condition that it be properly justified; no answer lacking its corresponding logical and mathematical development will be evaluated.

1. **Let the function** $f(x)=\dfrac{\ln(x+1)}{\sqrt{x+1}}$ **be defined on** $(-1,\infty)$**. Find:**
   - *a)* The asymptotes, the intervals of increase and decrease, and the global extrema of $f(x)$.
   - *b)* The range of $f(x)$ and sketch the graph of the function.
   - *c)* Consider $f_b(x)$ as the function $f(x)$ restricted to the interval $[b,\infty)$, where $b>-1$. Find the global maximum and minimum (if they exist) of $f_b(x)$.

   **0.4 points for part a); 0.3 points for part b); 0.3 points for part c)**

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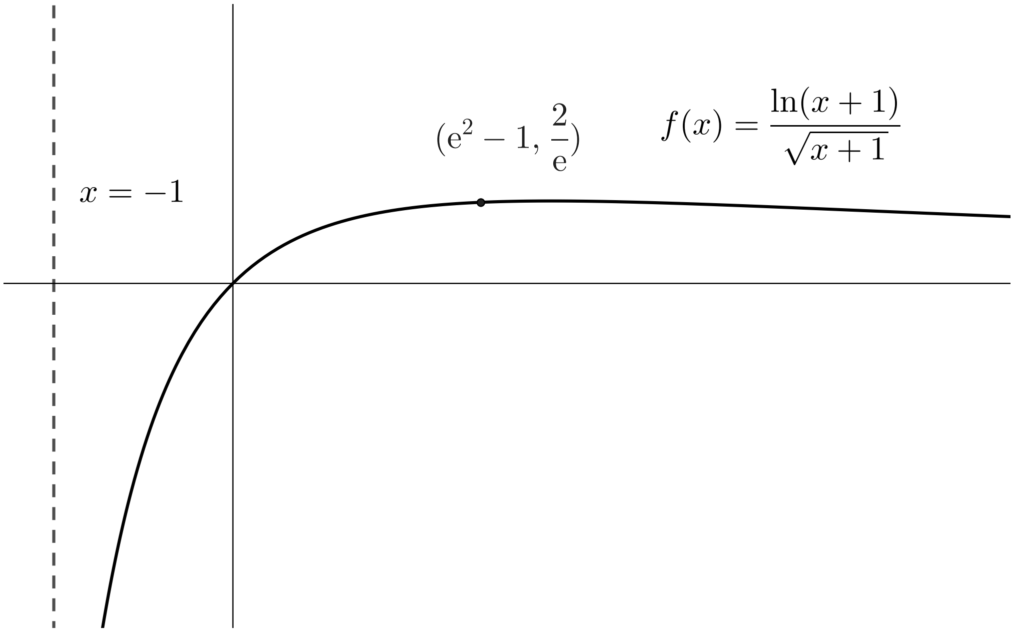
a) On the one hand, $f(x)$ is continuous on its entire domain, so we only need to compute the asymptotes at $-1^+$ and at $\infty$.
$\lim_{x\to -1^+} f(x)=\dfrac{-\infty}{0^+}=-\infty$, hence $f(x)$ has a vertical asymptote at $x=-1$ from the right.
$\lim_{x\to\infty} f(x)=\dfrac{\infty}{\infty}=$ (applying L'Hôpital's rule) $=\lim_{x\to\infty}\dfrac{1/(x+1)}{(x+1)^{-1/2}/2}=\lim_{x\to\infty}\dfrac{2}{\sqrt{x+1}}=0$, therefore $f(x)$ has a horizontal asymptote $y=0$ at $\infty$.
And since $f'(x)=\dfrac{(1/(x+1))\sqrt{x+1}-\ln(x+1)/(2\sqrt{x+1})}{x+1}=\dfrac{(x+1)^{-1/2}(2-\ln(x+1))}{2(x+1)}$,
the only critical point is obtained when $2-\ln(x+1)=0$, that is, $x=e^2-1$.
$f'(0)>0$, so $f'(x)>0$ if $x\in(-1,e^2-1)$, hence $f(x)$ is increasing on $(-1,e^2-1]$.
$f'(e^3-1)<0$, so $f'(x)<0$ if $x\in(e^2-1,\infty)$, hence $f(x)$ is decreasing on $[e^2-1,\infty)$. Obviously, $x=e^2-1$ is the global maximizer of $f(x)$.
On the other hand, $f(x)$ has no global minimizer.

b) From the above, the maximum value is $f(e^2-1)=\dfrac{2}{e}$. From this and the fact that $\lim_{x\to -1^+} f(x)=-\infty$, it follows that the range of $f(x)$ is $\left(-\infty,\frac{2}{e}\right]$, by the Intermediate Value Theorem for continuous functions. Thus, the graph of the function is shown at the end.

c) We have seen that $f(x)$ is increasing on $(-1,e^2-1]$, decreasing on $[e^2-1,\infty)$, $f(0)=0$, and $\lim_{x\to\infty} f(x)=0^+$.
Therefore, as long as $y=0$ is in the range of $f_b(x)$ or equivalently, $x=0$ belongs to the interval $[b,\infty)$ both global extrema will exist.
Thus, we must consider three cases:
i) if $b\le 0 \Longrightarrow \min(f_b)=f(b)$, $\max(f_b)=\dfrac{2}{e}$, so both the global maximum and minimum exist.
ii) if $0<b\le e^2-1 \Longrightarrow \min(f_b)$ does not exist, $\max(f_b)=\dfrac{2}{e}$.
iii) if $e^2-1\le b \Longrightarrow \min(f_b)$ does not exist, $\max(f_b)=f(b)$.
See, again, the graph of the function.


$x=-1$
$(\mathrm{e}^2-1, \frac{2}{\mathrm{e}})$
$f(x)=\frac{\ln(x+1)}{\sqrt{x+1}}$

2. **Given the function $y = f(x)$, implicitly defined by the equation $e^{x+2y} + e^{2x+y} = 2e^3$ in a neighborhood of the point $x = 1, y = 1$, find:**
   - *a*) The tangent line and the second-degree Taylor polynomial of $f$ at $a = 1$.
   - *b*) Sketch an approximate graph of $f(x)$ and $f^{-1}(x)$ near the point $x = 1$.
   - *c*) Compare both functions on the intervals $(1 - \delta, 1]$ and $[1, 1 + \delta)$.

     In each interval, will one of the two functions be greater, or will they be equal?

     Hint for b) and c): observe that $F(x, y) = F(y, x)$.

     **0.4 points for part a); 0.4 points for part b); 0.2 points for part c).**

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a) First of all, we observe that the point $(1, 1)$ satisfies the equation.
First, we compute the first derivative of the function:

$$(1 + 2y')e^{x+2y} + (2 + y')e^{2x+y} = 0$$

substituting $x = 1$ and $y(1) = 1$, we deduce that $y'(1) = f'(1) = -1$.
Hence, the equation of the tangent line is: $y = P_1(x) = 1 - (x - 1)$ or $y = -x + 2$.
Analogously, we compute the second derivative of the function:

$$[2y'' + (1 + 2y')^2]e^{x+2y} + [y'' + (2 + y')^2]e^{2x+y} = 0$$

substituting $x = 1, y(1) = 1$, and $y'(1) = -1$, it follows that:

$$[2y'' + 1]e^3 + [y'' + 1]e^3 = 0 \Longrightarrow y''(1) = f''(1) = -2/3$$

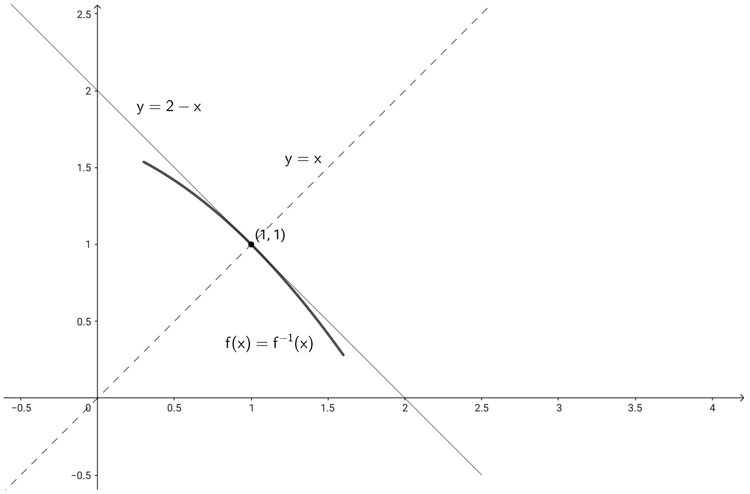
Therefore, the second-degree Taylor polynomial is: $P_2(x) = 1 - (x - 1) - \frac{1}{3}(x - 1)^2$.

b) Using the second-order Taylor polynomial, the graph of $f$ and its inverse near the point $x = 1$ will be approximately as shown in the figure at the end.

c) Both functions will be equal, since if the implicit function $y = f(x)$ is given by the equation $F(x, f(x)) = 2e^3$, then, by the given hint, it holds that $F(f(x), x) = 2e^3$.
Setting $f(x) = t$ and $x = f^{-1}(t)$, the previous equation can be written as:

$$F(t, f^{-1}(t)) = 2e^3$$

That is, the implicit function $f$ and its inverse $f^{-1}$ are the same function.



3. **Let $C(x) = 16 + 2x + x^2$ be the cost function and $p(x) = 20 - 2x$ be the inverse demand function of a monopolistic firm. Find:**
   *a*) The production level that maximizes profit.
   *b*) The production level that minimizes average cost.
   *c*) The government wants the firm to produce at the level found in part b). To achieve this, it proposes to subsidize each unit produced. What will be the minimum subsidy per unit required for the firm to agree to move from the level in part a) to that of part b), assuming it maximizes profits?
   **0.4 points for part a); 0.4 points for part b); 0.2 points for part c).**

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a) First, we calculate the profit function:

$$B(x) = (20 - 2x)x - (16 + 2x + x^2) = -3x^2 + 18x - 16$$

Computing the first and second derivatives of $B$:

$$B'(x) = -6x + 18; \quad B''(x) = -6 < 0$$

hence we see that $B$ has a unique critical point at $x^* = 3$ and, since $B$ is a concave function, this critical point is the unique global maximizer.

b) Since the average cost function is $\dfrac{C(x)}{x} = \dfrac{16}{x} + 2 + x$, its derivative is:

$$\left(\frac{C(x)}{x}\right)' = -\frac{16}{x^2} + 1 = 0 \Longleftrightarrow x = 4$$

Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{32}{x^3} > 0$, the function is convex and the critical point is a global minimizer.

c) $B(3) = -27 + 54 - 16 = 11; \quad B(4) = -48 + 72 - 16 = 8.$
Thus, the firm will require a total compensation of 3 monetary units.
Since the production level is 4 units, the subsidy per unit will be $3/4$.

4. **Let** $f(x) = \begin{cases} x^2 - 3x + 2, & x \leq 3 \\ 2x^2 + ax + b, & x > 3 \end{cases}$ **Find:**

*a*) State the Mean Value Theorem (Lagrange's Theorem) for a function $f(x)$ defined on the interval $[0, 4]$. Find the values of $a$ and $b$ such that the given function satisfies the hypotheses of this theorem.

*b*) State Bolzano's Theorem for a function $f(x)$ defined on the interval $[K, 4]$. Suppose now that for the given function $a = -6$, $b = 2$, and it is defined on the interval $[K, 4]$ where $K < 3$. For which values of $K$ are the hypotheses of the aforementioned theorem satisfied?

**0.5 points for part a); 0.5 points for part b).**

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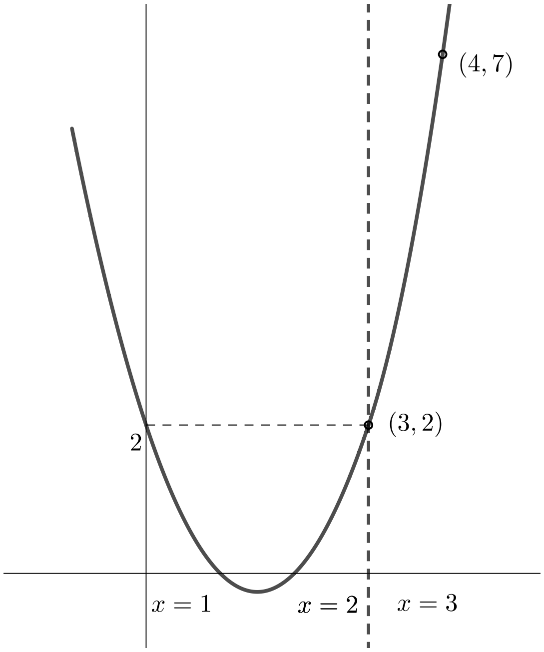
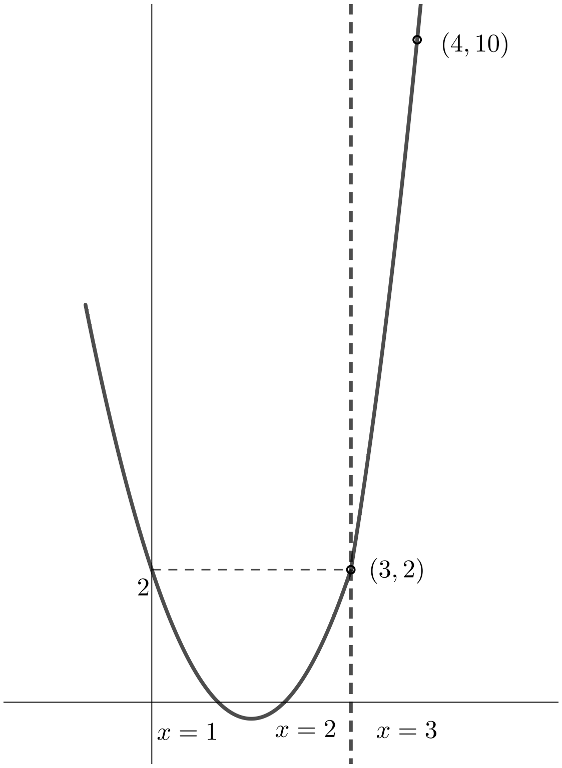
a) The hypotheses are that $f$ is continuous on $[0, 4]$ and differentiable on $(0, 4)$.
The thesis, or conclusion, is that there exists $c \in (0, 4)$ such that $f'(c) = (f(4) - f(0))/4$.
To achieve this, we first need to impose the continuity of $f$ at $x = 3$.
Since $\mathop{\text{lím}}_{x\to 3^+} f(x) = 18 + 3a + b$, and since $f(3) = 2 = \mathop{\text{lím}}_{x\to 3^-} f(x)$, it follows that the function will be continuous on $[0, 4]$ when: $3a + b = -16$.
On the other hand, assuming $f$ is continuous, the function will be differentiable at $x = 3$ when: $\mathop{\text{lím}}_{x\to 3^+} f'(x) = f'(3^-)$. Now we have:
i) $x > 3 \Longrightarrow \mathop{\text{lím}}_{x\to 3^+} f'(x) = \mathop{\text{lím}}_{x\to 3^+} (4x + a) = f'(3^+) = 12 + a$;
ii) $x < 3 \Longrightarrow f'(x) = 2x - 3 \Longrightarrow f'(3^-) = 3$.
Therefore, the hypotheses of Lagrange's theorem are satisfied when: $a = -9$, $b = 11$.



b) The hypotheses are that $f$ is continuous on $[K, 4]$ and that $f(K) \cdot f(4) < 0$.
The thesis, or conclusion, is that there exists $c \in (K, 4)$ such that $f(c) = 0$.
For the given values of $a$ and $b$, the function is continuous, since: $f(3) = 2 = \mathop{\text{lím}}_{x\to 3^+} f(x)$.
On the other hand, $f(4) > 0$, so for the hypotheses to hold, it must be that $f(K) < 0$.
For $x < 3$, we have $f(x) = (x - 1)(x - 2)$, hence $f(K) < 0$ when $1 < K < 2$.


(4, 10)
(3, 2)
2
x = 1
x = 2
x = 3

5. **Given $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{3}{1+x} - x$ and $g(x) = \sqrt{9-x}$, find:**

*a*) Sketch the set $A$ bounded by the graphs of $f(x)$, $g(x)$, and the lines $x = 0$ and $x = 9$.

*b*) Find, if they exist, the maximal and minimal elements, and the maximum and minimum of $A$.

*c*) Calculate the area of the given set, taking $\ln 10 \approx 2{,}35$.

*Hint for b*: The Pareto order is given by: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

**0.4 points for part a); 0.2 points for part b); 0.4 points for part c)**

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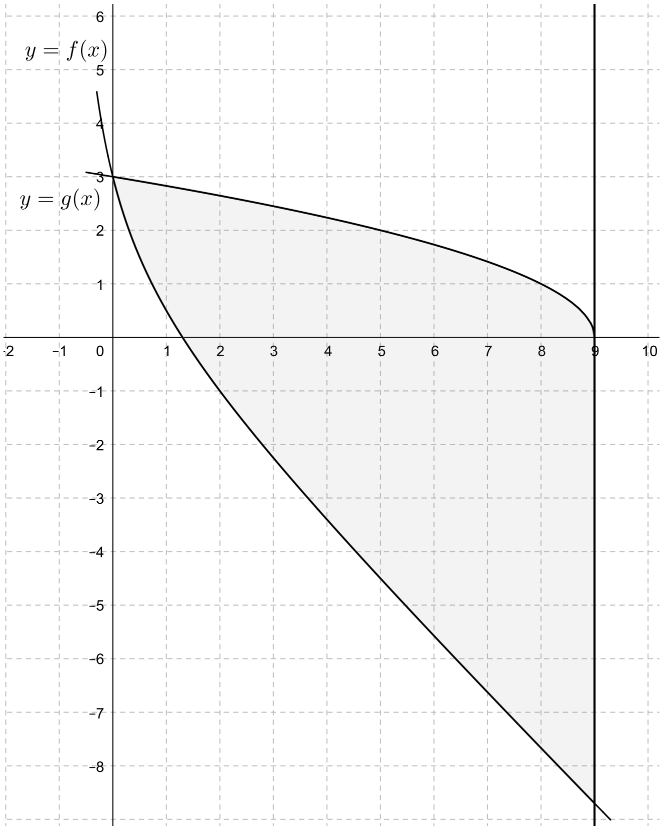
a) First, we observe that the graphs of $f(x)$ and $g(x)$ intersect the line $x = 9$ at the points $(9, -\frac{87}{10})$ and $(9, 0)$, respectively.

On the other hand, the graphs of $f(x)$ and $g(x)$ intersect at the point $(0, 3)$, and there cannot be another intersection to the right of this point because both functions are decreasing, $f(x)$ is convex, $g(x)$ is concave, and when they intersect the line $x = 9$, we have $f(9) = -\dfrac{87}{10} < 0 = g(9)$.

Therefore, $A = \{(x, y) : 0 \leq x \leq 9,\ f(x) \leq y \leq g(x)\}$.

Hence, the sketch of $A$ will be approximately as follows:



b) In this way, the Pareto order describes the set as follows:

maximum$(A)$ does not exist; maximal$(A) = \{(x, g(x)) : 0 \leq x \leq 9\}$.

minimum$(A)$ does not exist; minimal$(A) = \{(x, f(x)) : 0 \leq x \leq 9\}$.

c) First, from the relative positions of the functions, we know that:

$$\text{Area}(A) = \int_0^9 g(x)dx - \int_0^9 f(x)dx$$

Since $\displaystyle\int g(x)dx = \int \sqrt{9-x}dx = -\frac{2}{3}(9-x)^{3/2}$ and $\displaystyle\int f(x)dx = \int \left(\frac{3}{1+x} - x\right) dx = 3\ln(1+x) - x^2/2$, it follows by the Fundamental Theorem of Calculus (Barrow's Rule) that:

$\text{Area}(A) = \left[-\frac{2}{3}(9-x)^{3/2} - 3\ln(1+x) + x^2/2\right]_0^9 = 0 - 3\ln 10 + 81/2 + \frac{2}{3}27 + 0 - 0 =$

$= 40{,}5 + 18 - 3\ln 10 \approx 58{,}5 - 7{,}05 = 51{,}45$ area units.