1

(a) (5 points) Find the general solution of the difference equation

$$x_{t+2} - 2x_{t+1} + 4x_t = 0.$$

Note: $\sqrt{12} = 2\sqrt{3}$.

(b) (5 points) Find the general solution of the difference equation

$$x_{t+2} - 2x_{t+1} + 4x_t = t + 1.$$

(c) (5 points) Find the solution of the initial value problem

$$x_{t+2} - 2x_{t+1} + 4x_t = t + 1, \quad x_0 = \frac{2}{3},\ x_1 = \frac{2}{3}.$$

Solution:

(a) The characteristic equation is $r^2 - 2r + 4 = 0$, the solutions of which are $r_{1,2} = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\frac{2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$. The module is $|r_{1,2}| = \sqrt{1+3} = 2$ and the argument of r_1 is $\theta = \arctan\sqrt{3} = \frac{\pi}{3}$. Hence
$x_t = 2^t \left(C_1 \cos\left(\frac{\pi}{3}t\right) + C_2 \sin\left(\frac{\pi}{3}t\right)\right)$.

(b) Try a particular solution $At + B$, with unknown constants A and B which are determined by

$$A(t+2) + B - 2A(t+1) - 2B + 4At + 4B = t + 1,$$

obtaining $A = B = \frac{1}{3}$. Hence
$x_t = 2^t \left(C_1 \cos\left(\frac{\pi}{3}t\right) + C_2 \sin\left(\frac{\pi}{3}t\right)\right) + \frac{1}{3}t + \frac{1}{3}$.

(c) $\frac{2}{3} = C_1 + \frac{1}{3}$, hence $C_1 = \frac{1}{3}$
$x_1 = 2C_1 + C_2\sqrt{3} + \frac{2}{3}$, hence $C_2 = -\frac{1}{3\sqrt{3}}$
The particular solution is thus
$x_t = 2^t \left(\frac{1}{3} \cos\left(\frac{\pi}{3}t\right) - \frac{1}{3\sqrt{3}} \sin\left(\frac{\pi}{3}t\right)\right) + \frac{1}{3}t + \frac{1}{3}$

2

Let the matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ b & -2 & 2 \\ 2 & -2 & 3 \end{pmatrix},$$

where a and b are parameters.

(a) (5 points) Determine the eigenvalues of A.

(b) (5 points) For which values of the parameters a, b is the matrix A diagonalizable?

(c) (5 points) For the values $a = \frac{1}{2}$ and $b = 1$, determine a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

Solution:

(a) $p_A(\lambda) = (a - \lambda)(\lambda^2 - \lambda - 2) = 0$ has roots a, 2 and -1.

(b) Case (i): if $a \neq 2$ and $a \neq -1$, then A is diagonalizable, since A has 3 distinct eigenvalues.

Case (ii): if $a = 2$, then the eigenvalue 2 is double. The matrix $A - 2I_3$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ b & -4 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

which has rank 1 iff $b = 4$.

Case (iii): if $a = -1$, then the eigenvalue -1 is double. The matrix $A + I_3$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ b & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix}$$

which has rank 1 iff $b = 1$.

Summary: A is diagonalizable if $a \neq \pm 1$, or if $a = 2$ and $b = 4$ or if $a = -1$ and $b = 1$.

(c) $a = \frac{1}{2}$ means that we are in Case (i).

$u \in S(2)$ if $\begin{pmatrix} -\frac{3}{2} & 0 & 0 \\ 1 & -4 & 2 \\ 2 & -2 & 1 \end{pmatrix} u = 0$. That is, $x = 0$ and $z = 2y$. Letting $y = 1$, we obtain $(0, 1, 2)$.

$u \in S(\frac{1}{2})$ if $\begin{pmatrix} 0 & 0 & 0 \\ 1 & -\frac{5}{2} & 2 \\ 2 & -2 & \frac{5}{2} \end{pmatrix} u = 0$. That is $x - \frac{5}{2}y + 2z = 0$, $2x - 2y + \frac{5}{2}z = 0$. Eliminating x from both equations, $-3y - \frac{3}{2}z = 0$, or $y = \frac{z}{2}$. Letting $z = 1$ we obtain $(-\frac{3}{4}, \frac{1}{2}, 1)$.

$u \in S(-1)$ if $\begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix} u = 0$. That is, $(0, 2, 1)$.

Hence

$$P = \begin{pmatrix} 0 & -\frac{3}{4} & 0 \\ 1 & \frac{1}{2} & 2 \\ 2 & 1 & 1 \end{pmatrix}, \ D = \operatorname{diag}(2, \tfrac{1}{2}, -1).$$

3

Consider the following system of linear difference equations.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & -2 & 2 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Note that the matrix of the system is the matrix A of Problem 2 above when $a = \frac{1}{2}$ and $b = 1$.

(a) (5 points) Find the stationary or equilibrium point of the system. Study the stability of the equilibrium point (stable, locally stable, unstable, saddle point), finding the stable manifold in the case that the point is a saddle.

(b) (5 points) Determine the general solution of the system.

Solution:

(a) The steady state satisfies

$$\left\{ \begin{array}{rcllll} x & = & \frac{x}{2} & & & +1 \\ y & = & x & -2y & +2z & \\ z & = & 2x & -2y & +3z & +2 \end{array} \right.$$

The solution is $(2, -4, -7)$.

In Problem 2 we found that the matrix of coefficients has eigenvalues $\frac{1}{2}$, 2 and -1. The equilibrium point is unstable, but it is a saddle point. The stable manifold is $S(\frac{1}{2})$, which is the line of equations $\left\{ \begin{array}{c} x - \frac{5}{2}y + 2z = 0 \\ 2x - 2y + \frac{5}{2}z = 0 \end{array} \right.$.

(b) It has been found in Problem 2 that the matrix A is diagonalizable and the eigenvectors, thus the general solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = C_1 2^{-t} \begin{pmatrix} -\frac{3}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix} + C_2 2^t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + C_3 (-1)^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ -7 \end{pmatrix}.$$

where C_1, C_2 y C_3 are arbitrary constants.

4

Consider the ODE

$$2tx' + x = t^2, \quad t \neq 0.$$

(a) (5 points) Find the general solution.

(b) (5 points) Find the solution of the initial value problem

$$2tx' + x = t^2, \quad x(25) = 126.$$

What is the interval of definition?

Solution:

(a) Let us look for a solution of the form $x = uv$. We have $x' = u'v + uv'$. We obtain

$$2tv(t)u'(t) + 2tu(t)v'(t) + u(t)v(t) = t^2$$

We choose u so that

$$2tv'(t) + v(t) = 0$$

This is a separable equation. The solution is $v = \frac{1}{\sqrt{|t|}}$ for $t \neq 0$. With this choice, the ODE becomes $2\sqrt{|t|}u'(t) = t^2$ so $u = \frac{|t|^{5/2}}{5} + c_1$. The general solution is

$$x(t) = uv = \frac{t^2}{5} + \frac{c_1}{\sqrt{|t|}}, \quad t \neq 0$$

(b) We have

$$x(25) = 125 + \frac{c_1}{5} = 126$$

It is enough to take $c_1 = 5$. The solution is

$$x(t) = \frac{t^2}{5} + \frac{5}{\sqrt{t}}, \quad t \in (0, \infty)$$

5

(a) (5 points) Find the general solution of the ODE

$$x'' - x' - 6x = 0$$

(b) (5 points) Find the general solution of the ODE

$$x'' - x' - 6x = 30t + 5$$

(c) (5 points) Find the solution of the following initial value problem

$$x'' - x' - 6x = 30t + 5, \quad x(0) = 4 \quad x'(0) = 2$$

(d) (5 points) Find the solution $x(t)$ of the following initial value problem

$$x'' - x' - 6x = 0, \quad x(0) = 1 \quad \lim_{t\to\infty} x(t) = 0$$

Solution:

(a) The general solution is

$$c_1 e^{-2t} + c_2 e^{3t}$$

(b) We look for a particular solution of the form $x_p = A + Bt$. We have

$$x_p' = B \quad x_p'' = 0$$

Hence,

$$x'' - x' - 6x = -6A - B - 6Bt = 30t + 5$$

It is enough to take $A = 0$, $B = -5$. The general solution is

$$x(t) = c_1 e^{-2t} + c_2 e^{3t} - 5t$$

(c) We have

$$x'(t) = -2c_1 e^{-2t} + 3c_2 e^{3t} - 5$$

Thus,

$$\begin{aligned} x(0) &= 4 = c_1 + c_2 \\ x'(0) &= 2 = -5 - 2c_1 + 3c_2 \end{aligned}$$

We. obtain $c_1 = 1$, $c_2 = 3$. The solution is

$$c_1 e^{-2t} + c_2 e^{3t} - 5t$$

(d) The general solution is

$$c_1 e^{-2t} + c_2 e^{3t}$$

We must take $c_1 = 1$, $c_2 = 0$. The solution is $c_1 e^{-2t}$.

6

Consider the following system of ODE's

$$\begin{cases} x' &= -15 - 3x + 5y + xy \\ y' &= -2x + xy \end{cases}$$

(a) (5 points) Determine the stationary points.

(b) (5 points) Determine the stability of the stationary points.

Solution:

The stationary points are the solutions of the following system of equations

$$\begin{cases} 0 &= -15 - 3x + 5y + xy \\ 0 &= -2x + xy \end{cases}$$

The second equation may be written as $x(y-2) = 0$. So, either $x = 0$ or $y = 2$. The solutions are $(-5, 2)$ and $(0, 3)$. The Jacobian matrix of the system is

$$J(x, y) = \begin{pmatrix} y - 3 & x + 5 \\ y - 2 & x \end{pmatrix}$$

We see that

- $J(-5, 2) = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$. The eigenvalues are -1 and -4. Hence, $(-5, 2)$ is a stable node.
- $J(0, 3) = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$. The eigenvalues are $-\sqrt{5}$ and $\sqrt{5}$. Hence, $(0, 3)$ is a saddle point.