TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS

Sheet 6. Differential Equations (2)

Solutions

6-1. *Solve the following first order ODEs.*

(a) $\dot{x} = \frac{t^3}{x^3}$.
(b) $\dot{x} = \frac{x^3}{t^3}$.
(c) $y' = \frac{\sqrt{x+1}}{y^2}$, *with* $y(0) = 1$.
(d) $(t + 3x)\, dt + (t - 20)\, dx = 0$.
(e) $(2xy - \cos x)\, dx + (x^2 - 1)\, dy = 0$, *with* $y(0) = 0$.
(f) $\dot{x} + 2tx = \cos t e^{-t^2}$, *with* $x(0) = 0$.
(g) $\dot{x} + \frac{x}{t} = e^{-t^2}$.

Solution:

(a) (Separable) $x^4 - t^4 = C$;
(b) (Separable) $x^2 - t^2 = Ct^2x^2$;
(c) (Separable) $y^3 - 2\sqrt{(x+1)^3} = C$; from $y(0) = 1$ we get $C = -1$, so that $y^3 - 2\sqrt{(x+1)^3} + 1 = 0 \Rightarrow y(t) = \sqrt[3]{2\sqrt{(x+1)^3} - 1}$;
(d) (Not exact) An integration factor is

$$\mu(t) = e^{\int \frac{2}{t-20}\, dt} = e^{\ln (t-20)^2} = (t - 20)^2,$$

since

$$\frac{P_x - Q_t}{Q} = \frac{2}{t - 20} \qquad \text{is independent of } x.$$

multiplying the equation by $\mu(t)$ the equation is now exact. Hence, there is a function V such that $V_x = (t - 20)^3$; integrating we get $V(t, x) = (t - 20)^3x + g(t)$ and deriving with respect to t, $V_t = 3(t - 20)^2x + g'(t)$, which has to be equal to $(t - 20)^2(t + 3x)$, hence we get $g(t) = \int t(t-20)^2dt$. Taking parts $u = t$ and $dv = (t-20)^2$, one gets $g(t) = \frac{1}{12}(t-20)^3(3t+20)$. Hence, $V(t, x) = (t - 20)^3x + g(t) = (t - 20)^3x + \frac{1}{12}(t - 20)^3(3t + 20)$, and the solution is $(t - 20)^3(12x + 3t + 20) = C$;
(e) (Exact) $\sin x - y(x^2 - 1) = C$; from $y(0) = 0$ we get $C = 0$, so that $y(x) = \frac{\sin x}{x^2-1}$.
(f) (Linear) Take $\mu(t) = e^{\int 2t} = e^{t^2}$. We know (see the notes of the course)

$$x(t) = \frac{1}{e^{t^2}} \int \cos t e^{-t^2} e^{t^2}\, dt = e^{-t^2}(\sin t + C).$$

Now, the initial condition implies $C = 0$.
(g) (Linear) Take $\mu(t) = e^{\int \frac{1}{t}} = t$. We know (see the notes of the course)

$$x(t) = \frac{1}{t} \int te^{-t^2}\, dt = \frac{1}{t}\left(-\frac{1}{2}e^{-t^2} + C\right).$$

6-2. *The equation*

$$\dot{x} + a(t)x = b(t)x^n$$

is a Bernoulli equation. *It is a linear equation for* $n = 0$ *or* $n = 1$, *but it is not linear for* $n \neq 0, 1$. *Suppose that this is the case.*

(a) *Prove that the change of variable* $y = x^{1-n}$ *transforms the equation into a linear equation for* $y(t)$.
(b) *Solve* $\dot{x} + 2x = x^3$, $x(0) = 2$.

Solution:

(a)

$$\dot{y} = (1-n)x^{-n}\dot{x} = (1-n)x^{-n}(-ax + bx^n) = (1-n)(-ax^{1-n} + b) = (1-n)(-ay + b).$$

The linear equation for $y(t)$ is

$$\dot{y} + (1-n)a(t)y = (1-n)b(t),$$

that can be solved with the habitual technique, choosing $\mu(t) = e^{(1-n)\int a(t)\,dt}$.

(b) By the item above, the equation is transformed into

$$\dot{y} + 4y = 1$$

thus, $\mu(t) = e^{4t}$ and

$$(\dot{y} + 4y)e^{4t} = e^{4t}$$

implies, after integration

$$ye^{4t} = \int e^{4t}\,dt = \frac{1}{4}e^{4t} + C \quad \Rightarrow \quad y(t) = \frac{1}{4} + Ce^{-4t}.$$

Turning back to the original variable $y = x^{-2}$ we get

$$x^{-2}(t) = \frac{1}{4} + Ce^{-4t} \quad \Rightarrow \quad x(t) = \pm\frac{1}{\sqrt{\frac{1}{4} + Ce^{-4t}}}.$$

6-3. *Draw the phase diagrams of each of the following equations, find the equilibrium points and study their stability.*

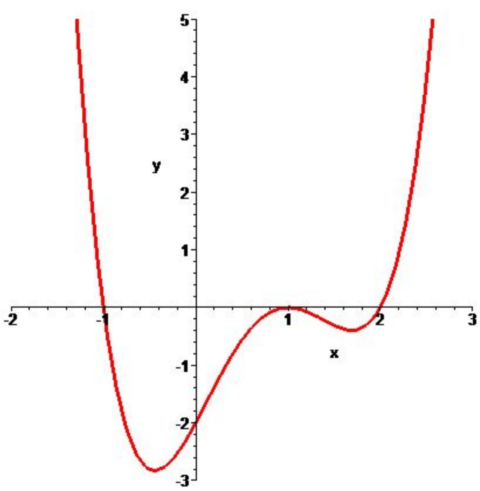
(a) $\dot{x} = g_1(x) = (x+1)(x-1)^2(x-2)$.

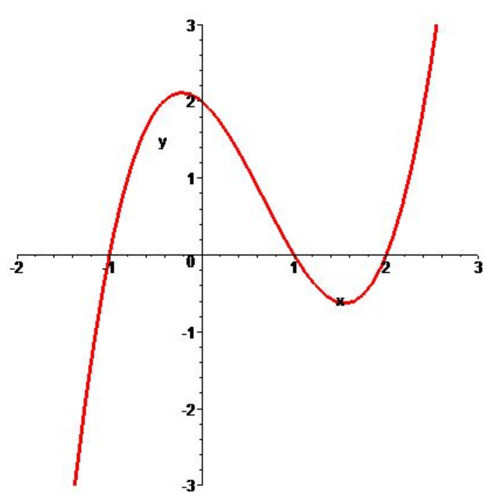
(b) $\dot{x} = g_2(x) = (x+1)(x-1)(x-2)$

Solution: Equilibrium points are constant solutions of the autonomous differential equation $\dot{x} = f(x)$, that is, solutions of $f(x) = 0$, because this implies $\dot{x} = 0$. In both cases -1, 1 and 2 are the only equilibrium points. To analyze the qualitative behavior of the equation we study the sign of the functions g_1 and g_2.

(a) The graph of g_1 shows that it is strictly positive in $(-\infty, 1)$ and strictly negative in $(-1, 2)$ (except at 1), hence any solution $x(t)$ with initial condition x_0 in the interval $(-\infty, -1)$ $((-1, 2))$ is strictly increasing (decreasing), thus $x(t)$ converges to -1. This equilibrium point is locally asymptotically stable. For the point 1, we see that any solution starting in the interval $(-1, 2)$ is strictly decreasing ($x_0 \neq 1$), thus for initial conditions $-1 < x_0 < 1$, the solution converges to -1, as we already know, and for initial conditions $1 < x_0 < 2$ it converges to 1; the equilibrium point is unstable. A similar analysis shows that 2 is also an unstable point.

(b) The points -1 and 2 are unstable and 1 is locally asymptotically stable: any solution with initial condition in the interval $(-1, 2)$ converges to 1.





Graph of $g_1(x) = (x+1)(x-1)^2(x-2)$ Graph of $g_2(x) = (x+1)(x-1)(x-2)$

6-4. *Suppose that the population y of a certain species of fish in a given area of the ocean is described by the logistic equation*

$$\dot{y} = r\left(1 - \frac{y}{K}\right)y.$$

The resource is used for food. Suppose that the rate at which fish are caught, $E(y)$, is proportional to the population y. Thus, we assume that $E(y) = Ey$, with E a positive constant. Then the logistic equation is replaced by

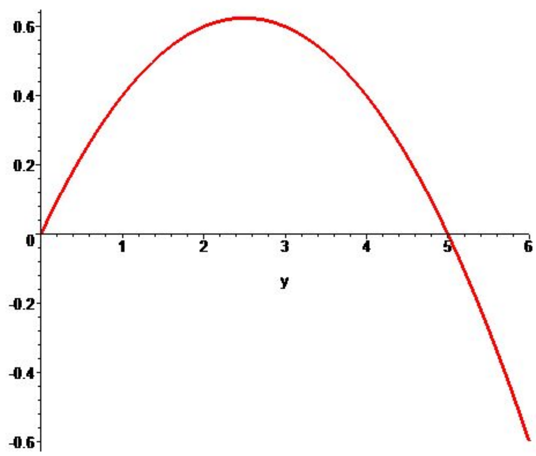
$$\dot{y} = r\left(1 - \frac{y}{K}\right)y - Ey.$$

This equation is known as Schaefer model

(a) *Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$, $y_2 > 0$;*
(b) *Show that y_1 is unstable and y_2 is asymptotically stable.*
(c) *A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is defined as Ey_2. Find Y as a function of the effort E and graph the function (it is known as the yield–effort curve).*
(d) *Determine E so as to maximize Y and thereby find the* maximum sustainable yield Y_m.

Solution:

(a) Solving $\dot{y} = 0$ we find $y_1 = 0$ and $y_2 = \left(1 - \frac{E}{r}\right)K > 0$.
(b) The function $f(y) = r\left(1 - \frac{y}{K}\right)y - Ey$ is positive in the interval $(0, y_2)$, hence the solutions starting in this interval are increasing and depart from 0 (the fish does not extinguish). On the other hand, f is negative in (y_2, ∞), hence the population of fish decreases to y_2 in this interval. See the figure below.



(c) $Y = Ey_2 = E\left(1 - \frac{E}{r}\right)K$. $Y(E)$ is a concave parabola with roots $E = 0$ and $E = r$, that is, the sustainable yield is 0 both when there is no captures or when the effort rate equals the intrinsic growth rate of the species.
(d) The effort that maximizes Y is obtained from (notice that Y is a concave function of E, hence critical points of Y are automatically global maximum of Y).

$$\frac{\partial Y}{\partial E} = \left(1 - 2\frac{E}{r}\right)K = 0,$$

hence

$$E_m = \frac{r}{2}$$

and the maximum sustainable yield is

$$Y_m = E_m y_2 = \frac{r}{2}\left(1 - \frac{E_m}{r}\right)K = K\frac{r}{4}.$$

6-5. *Consider the following growth model of Haavelmo. The output in the economy is given by a Cobb–Douglas production function with constant returns to scale*

$$Y = F(K, L) = AK^aL^{1-a}, \qquad 0 < a < 1,$$

where K is capital stock and L is level of employment; the constant $A > 0$, but we will take $A = 1$. It is supposed that the rate of growth of employment is not constant, but given by

(1) $$\frac{\dot{L}}{L} = \alpha - \beta\frac{L}{Y} = \alpha - \beta\frac{1}{Y/L}, \qquad \alpha, \beta > 0.$$

thus, the rate of growth of employment is an increasing function of per capita income (output), Y/L. The capital stock, K, is constant. *Plugging Y into* (1) *we get the ODE*

$$\dot{L} = \alpha L - \beta\frac{L^{1+a}}{K^a}, \qquad L(0) = L_0 > 0.$$

(a) *Find the equilibrium points (if any) and draw the phase diagram of the ODE.*
(b) *Study the asymptotic behavior of the solution.*

Solution:

(a) $\dot{L} = 0$ admits two solutions: $L_1^0 = 0$ and $L_2^0 = \left(\frac{\alpha}{\beta}\right)^{1/a} K$, where K is the constant stock of capital. The trajectories grow when $L(\alpha - \beta\frac{L^a}{K^a}) > 0$ and decrease otherwise, that is, when $L > 0$, they grow when $L(t) < L_2^0$ and decreases when $L(t) > L_2^0$.
(b) By the item above, $L = 0$ is unstable and every trajectory with $L(0) > 0$ converges to $L_2^0 = \left(\frac{\alpha}{\beta}\right)^{1/a} K$. This is the long run value of the labor force in this economy.

6-6. *Five college students with the flu return after Christmas Holidays to an isolated campus of 2500 students. If the rate at which this virus spreads is proportional to the number of infected students y and to the number not infected* $2500 - y$*, solve the initial value problem*

$$\dot{y} = ky(2500 - y), \qquad y(0) = 5$$

to find the number of infected students after t days if 25 students have the virus after one day. How many students have the flu after five days? Determine the number of days required for half the campus to be infected.

Solution: The equation

$$\dot{y} = ky(a - y)$$

is separable ($a = 2500$). To find the solution we separate variables and integrate as follows

$$\frac{dy}{y(a-y)} = k\,dt,$$

$$\int \frac{dy}{y(a-y)} = k\,dt = kt + C', \quad C' \text{ constant}$$

$$\int \frac{1}{ay} + \frac{1}{a(a-y)}\,dy = kt + C',$$

$$\ln\frac{y}{a-y} = a(kt + C'),$$

(2) $$\frac{y}{a-y} = Ce^{akt}, \quad \text{where } C = e^{aC'},$$

from which

$$y(t) = \frac{aCe^{akt}}{1 + Ce^{akt}} = \frac{2500Ce^{2500kt}}{1 + Ce^{2500kt}}.$$

The initial condition gives from (2) $C = \frac{5}{2500-5} = \frac{1}{499} \approx 0.002004$ and $y(1) = 25$ is again used in (2) to determine the constant k:

$$\frac{25}{2500 - 25} = \frac{1}{499}e^{2500k} \Rightarrow k = \frac{1}{2500}\ln\frac{499}{99} \approx 0.000647.$$

The solution is thus,

$$y(t) = \frac{5.01002e^{1.61749t}}{1 + 0.002004e^{1.61749t}}.$$

Now,

$$y(5) \approx 2167 \quad \text{students.}$$

For half the campus to be infected the time elapsed must satisfy $y(\hat{t}) = 1250$. To find this $\hat{t}$, we use again (2) to get

$$\frac{1250}{2500 - 1250} = Ce^{ak\hat{t}} \qquad \Rightarrow \hat{t} = \frac{1}{ak} \ln \frac{1}{C} \approx 3.84 \text{ days.}$$