Topic 2: Difference Equations (I)

1. Introduction

In this chapter we shall consider systems of equations where each variable has a time index $t = 0, 1, 2, \ldots$ and variables of different time–periods are connected in a non–trivial way. Such systems are called *systems of difference equations* and are useful to describe *dynamical systems with discrete time*. The study of dynamics in economics is important because it allows to drop out the (static) assumption that the process of economic adjustment inevitable leads to an equilibrium. In a dynamic context, this stability property has to be checked, rather than assumed away.

Let time be a discrete denoted $t = 0, 1, \ldots$. A function $X : \mathbb{N} \longrightarrow \mathbb{R}^n$ that depends on this variable is simply a sequence of vectors of n dimensions

$$X_0, X_1, X_2, \ldots$$

If each vector is connected with the previous vector by means of a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as

$$X_{t+1} = f(X_t), \qquad t = 0, 1, \ldots,$$

then we have a *system of first–order difference equations*. In the following definition, we generalize the concept to systems with longer time lags and that can include t explicitly.

Definition 1.1. A kth order discrete system of difference equations is an expression of the form

$$X_{t+k} = f(X_{t+k-1}, \ldots, X_t, t), \qquad t = 0, 1, \ldots, \tag{1.1}$$

where every $X_t \in \mathbb{R}^n$ and $f : \mathbb{R}^{nk} \times [0, \infty) \longrightarrow \mathbb{R}^n$. The system is

- *autonomous*, if f does not depend on t;
- *linear*, if the mapping f is linear in the variables $(X_{t+k-1}, \ldots, X_t)$;
- *of first order*, if $k = 1$.

Definition 1.2. A sequence $\{X_0, X_1, X_2, \ldots\}$ obtained from the recursion (1.1) with initial value X_0 is called a trajectory, orbit or path of the dynamical system from X_0.

In what follows we will write x_t instead of X_t if the variable X_t is a scalar.

Example 1.3. [Geometrical sequence] Let $\{x_t\}$ be a scalar sequence, $x_{t+1} = qx_t$, $t = 0, 1, \ldots$, with $q \in \mathbb{R}$. This a first–order, autonomous and linear difference equation. Obviously $x_t = q^t x_0$. Similarly, for arithmetic sequence, $x_{t+1} = x_t + d$, with $d \in \mathbb{R}$, $x_t = x_0 + td$.

Example 1.4.

- $x_{t+1} = x_t + t$ is linear, non–autonomous and of first order;
- $x_{t+2} = -x_t$ is linear, autonomous and of second order;
- $x_{t+1} = x_t^2 + 1$ is non–linear, autonomous and of first order;

Example 1.5. [Fibonacci numbers (1202)] "How many pairs of rabbits will be produced in a year, beginning with a single pair, if every month each pair bears a new pair which becomes productive from the second month on?". With x_t denoting the pairs of rabbits in month t, the problem leads to the following recursion

$$x_{t+2} = x_{t+1} + x_t, \qquad t = 0, 1, 2, \dots, \text{ with } x_0 = 1 \text{ and } x_1 = 1.$$

This is an autonomous and linear second–order difference equation.

2. Systems of first order difference equations

Systems of order $k > 1$ can be reduced to first order systems by augmenting the number of variables. This is the reason we study mainly first order systems. Instead of giving a general formula for the reduction, we present a simple example.

Example 2.1. Consider the second–order difference equation $y_{t+2} = g(y_{t+1}, y_t)$. Let $x_{1,t} = y_{t+1}$, $x_{2,t} = y_t$, then $x_{2,t+1} = y_{t+1} = x_{1,t}$ and the resulting first order system is

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} g(x_{1,t}, x_{2,t}) \\ x_{1,t} \end{pmatrix}.$$

If we denote $X_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$, $f(X_t) = \begin{pmatrix} g(X_t) \\ x_{1,t} \end{pmatrix}$, then the system can be written $X_{t+1} = f(X_t)$.

For example, $y_{t+2} = 4y_{t+1} + y_t^2 + 1$ can be reduced to the first order system

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 4x_{1,t} + x_{2,t}^2 + 1 \\ x_{1,t} \end{pmatrix},$$

and the Fibonacci equation of Example 1.5 is reduced to

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} x_{1,t} + x_{2,t} \\ x_{1,t} \end{pmatrix},$$

For a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we shall use the following notation: f^t denotes the t–fold composition of f, i.e. $f^1 = f$, $f^2 = f \circ f$ and, in general, $f^t = f \circ f^{t-1}$ for $t = 1, 2, \dots$. We also define f^0 as the identity function, $f^0(X) = X$.

Theorem 2.2. *Consider the autonomous first order system $X_{t+1} = f(X_t)$ and suppose that there exists some subset D such that for any $X \in D$, $f(X) \in D$. Then, given any initial condition $X_0 \in D$, the sequence $\{X_t\}$ is given by*

$$X_t = f^t(X_0).$$

Proof. Notice that

$$\begin{aligned} X_1 &= f(X_0), \\ X_2 &= f(X_1) = f(f(X_0)) = f^2(X_0), \\ &\vdots \\ X_t &= f(f \cdots f(X_0) \cdots)) = f^t(X_0). \end{aligned}$$

□

The theorem provides the current value of X, X_t, in terms of the initial value, X_0. We are interested what is the behavior of X_t in the future, that is, in the limit

$$\lim_{t\to\infty} f^t(X_0).$$

Generally, we are more interested in this limit that in the analytical expression of X_t. Nevertheless, there are some cases where the solution can be found explicitly, so we can study the above limit behavior quite well. Observe that if the limit exists, $\lim_{t\to\infty} f^t(X_0) = X^0$, say, and f is continuous

$$f(X^0) = f(\lim_{t\to\infty} f^t(X_0)) = \lim_{t\to\infty} f^{t+1}(X_0) = X^0,$$

hence the limit X^0 is a *fixed point* of map f. This is the reason fixed points play a distinguished role in dynamical systems.

Definition 2.3. A point $X^0 \in D$ is called a fixed point of the autonomous system f if, starting the system from X^0, it stays there:

$$\text{If } X_0 = X^0, \text{ then } X_t = X^0, \qquad t = 1, 2, \ldots.$$

Obviously, X^0 is also a fixed point of map f. A fixed point is also called *equilibrium*, *stationary point*, or *steady state*.

Example 2.4. In Example 1.3 ($x_{t+1} = qx_t$), if $q = 1$, then every point is a fixed point; if $q \neq 1$, then there exists a unique fixed point: $x^0 = 0$. Notice that the solution $x_t = q^t x_0$ has the following limit ($x_0 \neq 0$) depending the value of q.

$$\begin{aligned} -1 < q < 1 &\Rightarrow \lim_{t\to\infty} q^t x_0 = 0, \\ q = 1 &\Rightarrow \lim_{t\to\infty} q^t x_0 = x_0, \\ q \leq -1 &\Rightarrow \text{ the sequence oscillates between } + \text{ and } - \text{ and the limit does not exist} \end{aligned}$$

In Example 1.5, $x^0 = 0$ is the unique fixed point. Consider now the difference equation $x_{t+1} = x_t^2 - 6$. Then, the fixed points are the solutions of $x = x^2 - 6$, that is, $x^0 = -2$ and $x^0 = 3$.

In the following definitions, $\|X - Y\|$ stands for the Euclidean distance between X and Y. For example, if $X = (1, 2, 3)$ and $Y = (3, 6, 7)$, then

$$\|X - Y\| = \sqrt{(3-1)^2 + (6-2)^2 + (7-3)^2} = \sqrt{36} = 6.$$

Definition 2.5.

- A fixed point X^0 is called stable if for any close enough initial state X_0, the resulting trajectory $\{X_t\}$ exists and stays close forever to X^0, that is, for any positive real ε, there exists a positive real $\delta(\varepsilon)$ such that if $\|X_0 - X^0\| < \delta(\epsilon)$, then $\|X_t - X^0\| < \varepsilon$ for every t.
- A stable fixed point X^0 is called locally asymptotically stable (l.a.s.) if the trajectory $\{X_t\}$ starting from any initial point X_0 close to enough to X^0, converges to the fixed point.
- A stable fixed point is called globally asymptotically stable (g.a.s.) if any trajectory generated by any initial point X_0 converges to it.
- A fixed point is unstable if it is not stable or asymptotically stable.

Remark 2.6.

- If X^0 is stable, but not l.a.s., $\{X_t\}$ need not approach X^0.
- A g.a.s. fixed point is necessarily unique.
- If X^0 is l.a.s., then small perturbations around X^0 decay and the trajectory generated by the system returns to the fixed point as the time grows.

Definition 2.7. Let P be an integer larger than 1. A series of vectors $X_0, X_1, \ldots, X_{P-1}$ is called a P–period cycle of system f if a trajectory starting from X_0 goes through $X_1, \ldots, X_{P-1}$ and returns to X_0, that is

$$X_{t+1} = f(X_t), \qquad t = 0, 1, \ldots, P-1, \qquad X_P = X_0.$$

Observe that the series of vectors $X_0, X_1, \ldots, X_P$ repeats indefinitely in the trajectory,

$$\{X_t\} = \{X_0, X_1, \ldots, X_{P-1}, X_0, X_1, \ldots, X_{P-1}, \ldots\}.$$

For this reason, the trajectory itself is called a P–cycle.

Example 2.8. In Example 1.3 ($x_{t+1} = qx_t$) with $q = -1$ all the trajectories contains 2–cycles, because a typical path is

$$\{x_0, -x_0, x_0, -x_0, \ldots\}.$$

Example 2.9. In Example 1.4 where $y_{t+2} = -y_t$, to find the possible cycles of the equation, first we write it as first order system using Example 2.1, to obtain

$$X_{t+1} = \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} -x_{2,t} \\ x_{1,t} \end{pmatrix} \equiv f(X_t).$$

Let $X_0 = (2, 4)$. Then

$$\begin{aligned} X_1 &= f(X_0) = (-4, 2), \\ X_2 &= f(X_1) = (-2, -4), \\ X_3 &= f(X_2) = (4, -2), \\ X_4 &= f(X_3) = (2, 4) = X_0. \end{aligned}$$

Thus, a 4–cycle appears starting at X_0. In fact, any trajectory is a 4–cycle.

3. First order linear difference equations

The linear equation is of the form

$$x_{t+1} = ax_t + b, \qquad x_t \in \mathbb{R}, \qquad a, b \in \mathbb{R}. \tag{3.1}$$

Consider first the case $b = 0$ (homogeneous case). Then, by Theorem 2.2 the solution is $x_t = a^t x_0$, $t = 0, 1, \ldots$. Consider now the non–homogeneous case, $b \neq 0$. Let us find the fixed points of the equation. They solve (see Definition 2.3)

$$x^0 = ax^0 + b,$$

hence there is no fixed point if $a = 1$. However, if $a \neq 1$, the unique fixed point is

$$x^0 = \frac{b}{1-a}.$$

Define now $y_t = x_t - x^0$ and replace $x_t = y_t + x^0$ into (3.1) to get

$$y_{t+1} = ay_t,$$

hence $y_t = a^t y_0$. Returning to the variable x_t we find that the solution of the linear equation is

$$\begin{aligned} x_t &= x^0 + a^t(x_0 - x^0) \\ &= \frac{b}{1-a} + a^t\left(x_0 - \frac{b}{1-a}\right). \end{aligned}$$

Theorem 3.1. *In* (3.1), *the fixed point* $x^0 = \frac{b}{1-a}$ *is g.a.s. if and only if* $|a| < 1$.

Proof. Notice that $\lim_{t\to\infty} a^t = 0$ iff $|a| < 1$ and hence $\lim_{t\to\infty} x_t = \lim_{t\to\infty} x^0 + a^t(x_0 - x^0) = x^0$ iff $|a| < 1$, independently of the initial x_0. □

The convergence is monotonous if $0 < a < 1$ and oscillating if $-1 < a < 0$.

Example 3.2 (A Multiplier–Accelerator Model of Growth)**.** Let Y_t denote national income, I_t total investment, and S_t total saving—all in period t. Suppose that savings are proportional to national income, and that investment is proportional to the change in income from period t to $t+1$. Then, for $t = 0, 1, 2, \ldots$,

$$\begin{aligned} S_t &= \alpha Y_t, \\ I_{t+1} &= \beta(Y_{t+1} - Y_t), \\ S_t &= I_t. \end{aligned}$$

The last equation is the equilibrium condition that saving equals investment in each period. Here $\beta > \alpha > 0$. We can deduce a difference equation for Y_t and solve it as follows. From the first and third equation, $I_t = \alpha Y_t$, and so $I_{t+1} = \alpha Y_{t+1}$. Inserting these into the second equation yields $\alpha Y_{t+1} = \beta(Y_{t+1} - Y_t)$, or $(\alpha - \beta)Y_{t+1} = -\beta Y_t$. Thus,

$$Y_{t+1} = \frac{\beta}{\beta - \alpha} Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right) Y_t, \qquad t = 0, 1, 2, \ldots.$$

The solution is

$$Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right)^t Y_0, \qquad t = 0, 1, 2, \ldots.$$

Thus, Y grows at the constant proportional rate $g = \alpha/(\beta - \alpha)$ each period. Note that $g = (Y_{t+1} - Y_t)/Y_t$.

Example 3.3 (A Cobweb Model)**.** Consider a market model with a single commodity where producer's output decision must be made one period in advance of the actual sale—such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output. Let us assume that the output decision in period t is based in the prevailing price P_t, but since this output will no be available until period $t+1$, the supply function is lagged one period,

$$Q_{s,t+1} = S(P_t).$$

Suppose that demand at time t is determined by a function that depends on P_t,

$$Q_{d,t+1} = D(P_t).$$

Supposing that functions S and D are linear and that in each time period the market clears, we have the following three equations

$$\begin{aligned} Q_{d,t} &= Q_{s,t}, \\ Q_{d,t+1} &= \alpha - \beta P_{t+1}, \qquad \alpha, \beta > 0, \\ Q_{s,t+1} &= -\gamma + \delta P_t, \qquad \gamma, \delta > 0. \end{aligned}$$

By substituting the last two equations into the first the model is reduced to the difference equation for prices

$$P_{t+1} = -\frac{\delta}{\beta} P_t + \frac{\alpha + \gamma}{\beta}.$$

The fixed point is $P^0 = (\alpha + \gamma)/(\beta + \delta)$, which is also the equilibrium price of the market, that is, $S(P^0) = D(P^0)$. The solution is

$$P_{t+1} = P^0 + \left(-\frac{\delta}{\beta}\right)^t (P_0 - P^0).$$

Since $-\delta/\beta$ is negative, the solution path is oscillating. It is this fact which gives rise to the cobweb phenomenon. There are three oscillations patterns: it is *explosive* if $\delta > \beta$ (S steeper than D), *uniform* if $\delta = \beta$, and *damped* if $\delta < \beta$ (S flatter than D). The three possibilities are illustrated in the graphics below. The demand is the downward–slopping line, with slope $-\beta$. The supply is the upward–slopping line, with slope δ. When $\delta > \beta$, as in Figure 3, the interaction of demand and supply will produce an explosive oscillation as follows: Given an initial price P_0, the quantity supplied in the next period will be $Q_1 = S(P_0)$. In order to clear the market, the quantity demanded in period 1 must be also Q_1, which is possible if and only if price is set at the level of P_1 given by the equation $Q_1 = D(P_1)$. Now, via the S curve, the price P_1 will lead to $Q_2 = S(P_1)$ as the quantity supplied in period 2, and to clear the market, price must be set at the level of P_2 according to the demand curve. Repeating this reasoning, we can trace out a "cobweb" around the demand and supply curves.

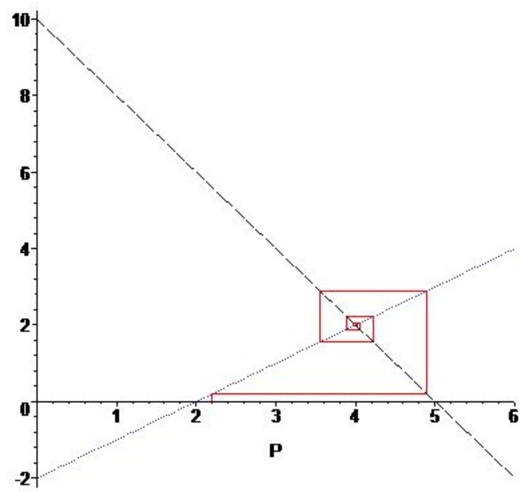

FIGURE 1. Cobweb diagram with damped oscillations

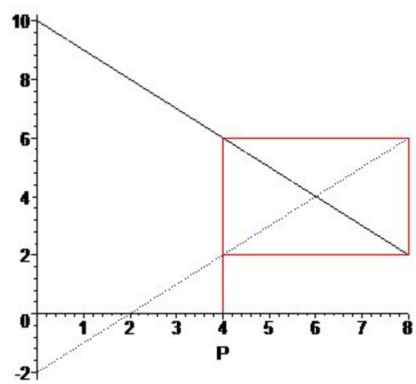

FIGURE 2. Cobweb diagram with uniform oscillations

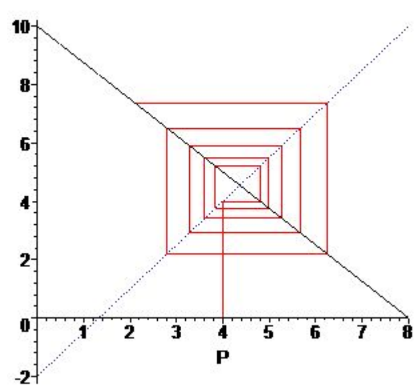

FIGURE 3. Cobweb diagram with explosive oscillations