University Carlos III of Madrid **Departament of Economics**
Game Theory: Problem set 5: Bayesian Games. Solutions.

Problem 1: Consider a Cornout duopoly which operates in a market with the following inverse demand function

$$P(Q) = \begin{cases} 90 - Q & \text{if } Q \leq 90, \\ 0 & \text{if } Q > 90. \end{cases}$$

where $Q = q_1 + q_2$ is the total output in the market. The cost of firm 2 is $c_2(q_2) = 9q_2$ with probability 1/3 and $c_2(q_2) = 27q_2$ with probability 2/3. The cost of firm 1 is $c_1(q_1) = 18q_1$. Firm 2 knows its own cost, but firm 1 only knows the types of costs of firm 2 and its probabilities. The above description is common knowledge.

1. Represent the above situation as a Bayesian Game. That is, describe the set set of players, their types, the set of strategies, their beliefs and their utilities.

   Solution: *There are two players,* $N = \{1, 2\}$*. Their types are* $T_1 = \{r\}$*,* $T_2 = \{c_l, c_h\}$ *where* c_l *represents the situation when firm 2 knows that its marginal cost is* $c_l = 9$ *and* c_h *represents the situation when firm 2 knows that its marginal cost is* $c_h = 27$*. The sets of strategies are* $S_1 = [0, \infty)$ *and* $S_2 = [0, \infty) \times [0, \infty) = \{(s_l, s_h) : s_l, s_h \in [0, \infty)\}$*. Here* s_l *(resp.* s_h*) represents the strategy of firm 2 when it knows that its marginal cost is* $c_l = 9$ *(resp.* $c_h = 27$*). The beliefs of the players are*

   $$p_1(c_l|r) = \frac{1}{3}; \quad p_1(c_h|r) = \frac{2}{3}, \quad p_2(r|a_l) = p_2(r|a_h) = 1$$

   The utilities of the players are the following.

   $$\begin{aligned} u_1(q_1, q_l, q_h|r) &= \frac{2}{3}(72 - q_1 - q_h)q_1 + \frac{1}{3}(72 - q_1 - q_l)q_1 \\ u_l(q_1, q_l|c_l) &= (81 - q_1 - q_l)q_l \\ u_h(q_1, q_h|c_h) &= (63 - q_1 - q_h)q_h \end{aligned}$$

2. Compute the Bayesian Equilibrium and the benefits of the firms in this equilibrium.

   Solution: *The best response of type* c_l *firm 2 is the solution of the following maximization problem*

   $$\max_{q_l} q_l(81 - q_1 - q_l)$$

   The solution is

   $$q_l = \frac{81 - q_1}{2} \tag{0.1}$$

   The best response of type c_h *firm 2 is the solution of the following maximization problem*

   $$\max_{q_h} q_h(63 - q_1 - q_h)$$

   The solution is

   $$q_h = \frac{63 - q_1}{2} \tag{0.2}$$

   The best response of firm 1 is the solution of the following maximization problem

   $$\max_{q_1} \frac{1}{3}(72 - q_1 - q_l)\, q_1 + \frac{2}{3}(72 - q_1 - q_h)\, q_1$$

   The solution is

   $$q_1 = \frac{216 - q_l - 2q_h}{6} \tag{0.3}$$

The Bayesian–Nash equilibrium is the solution to equations (0.1), (0.2) *and* (0.3). *We obtain,*

$$q_1^* = 25, \quad q_l^* = 28, \quad q_h^* = 19$$

The benefits are,

$$\Pi_1^* = 625 \quad \Pi_l^* = 784 \quad \Pi_h^* = 361$$

3. Suppose now that firm 1 knows that the costs of firm 2 is $c_2(q_2) = 9q_2$. Compute the Nash equilibrium and the benefits of the firms in this equilibrium.

   Solution: *The best response of type c_l firm 2 is the solution of the following maximization problem*

   $$\max_{q_l} q_l(81 - q_1 - q_l)$$

   The solution is

   $$q_l = \frac{81 - q_1}{2} \tag{0.4}$$

   The best response of firm 1 is the solution of the following maximization problem

   $$\max_{q_1} (72 - q_1 - q_l)\, q_1$$

   The solution is

   $$q_1 = \frac{72 - q_l}{2} \tag{0.5}$$

   The Bayesian–Nash equilibrium is the solution to equations (0.4) *and* (0.5). *We obtain,*

   $$\bar{q}_1 = 21, \quad \bar{q}_l = 30$$

   The benefits are,

   $$\bar{\Pi}_1 = 441 \quad \bar{\Pi}_l = 900$$

4. Suppose now that firm 1 knows that the costs of firm 2 is $c_2(q_2) = 27q_2$. Compute the Nash equilibrium and the benefits of the firms in this equilibrium.

   Solution: *The best response of type c_h firm 2 is the solution of the following maximization problem*

   $$\max_{q_h} q_h(63 - q_1 - q_h)$$

   The solution is

   $$q_h = \frac{63 - q_1}{2} \tag{0.6}$$

   The best response of firm 1 is

   $$q_1 = \frac{72 - q_h}{2}$$

   The Bayesian–Nash equilibrium is the solution to equations (0.4) *and* (0.6). *We obtain,*

   $$\tilde{q}_1 = 27, \quad \tilde{q}_h = 18$$

   The benefits are,

   $$\tilde{\Pi}_1 = 729 \quad \tilde{\Pi}_h = 324$$

5. In view of the above computation there is one type of firm 2 which prefers complete information and one type that prefers the situation with incomplete information. Identify those types.

   Solution: *Type c_l of firm 2 produces $q_l = 28$ and obtains a profit of $\Pi_l = 784$ with incomplete information and produces $q_l = 30$ and obtains a profit of $\Pi_l = 900$ with complete information. It* *prefers the situation with complete information. This firm would benefit if it could credibly inform firm 1 that the the cost of firm 2 is low* c_l*.*

Type c_h *of firm 2 produces* $q_h = 19$ *and obtains a profit of* $\Pi_h = 361$ *with incomplete information and produces* $q_h = 18$ *and obtains a profit of* $\Pi_h = 324$ *with complete information. It prefers the situation with incomplete information. This firm prefers that firm 1 is either not informed or (even better) that firm 1 believes (wrongly) that the the cost of firm 2 is low* c_l*.*

Finally, firm 1 prefers the situation with incomplete information when it faces firm 2 with cost c_l*. And it prefers the situation with complete information when it faces firm 2 with cost* c_h*.*

Complete information c_l		**Incomplete information**		**Complete information** c_h
$\bar{q}_1 = 21$	$<$	$q_1^* = 25$	$<$	$\tilde{q}_1 = 27$
$\bar{\Pi}_1 = 441$	$<$	$\Pi_1^* = 625$	$<$	$\tilde{\Pi}_1 = 729$
$\bar{q}_l = 30$	$>$	$q_l^* = 28$		
$\bar{\Pi}_l = 900$	$>$	$\Pi_l^* = 784$		
		$q_h^* = 19$	$>$	$\tilde{q}_h = 18$
		$\Pi_h^* = 361$	$>$	$\tilde{\Pi}_h = 324$

Problem 2: Consider a Cornout duopoly which operates in a market with the following inverse demand function

$$P(Q) = \begin{cases} 60 - Q & \text{if } Q \leq 60, \\ 0 & \text{if } Q > 60. \end{cases}$$

where $Q = q_1 + q_2$ is the total output in the market. The cost of firm 2 is $c_2(q_2) = 12q_2$ with probability 1/4 and $c_2(q_2) = 24q_2$ with probability 3/4. The cost of firm 1 is $c_1(q_1) = 18q_1$. Firm 2 knows its own cost, but firm 1 only knows the types of costs of firm 2 and its probabilities. The above description is common knowledge.

1. Represent the above situation as a Bayesian Game. That is, describe the set set of players, their types, the set of strategies, their beliefs and their utilities.

   Solution: *There are two players,* $N = \{1, 2\}$*. Their types are* $T_1 = \{r\}$*,* $T_2 = \{c_l, c_h\}$ *where* c_l *represents the situation when firm 2 knows that its marginal cost is* $c_l = 12$ *and* c_h *represents the situation when firm 2 knows that its marginal cost is* $c_h = 24$*. The sets of strategies are* $S_1 = [0, \infty)$ *and* $S_2 = [0, \infty) \times [0, \infty) = \{(s_l, s_h) : s_l, s_h \in [0, \infty)\}$*. Here* s_l *(resp.* s_h*) represents the strategy of firm 2 when it knows that its marginal cost is* $c_l = 12$ *(resp.* $c_h = 24$*). The beliefs of the players are*

   $$p_1(c_l|r) = \frac{1}{4}; \quad p_1(c_h|r) = \frac{3}{4}, \quad p_2(r|a_l) = p_2(r|a_h) = 1$$

   The utilities of the players are the following.

   $$\begin{aligned} u_1(q_1, q_2|r) &= \frac{1}{4}(42 - q_1 - q_l)q_1 + \frac{3}{4}(42 - q_1 - qh2)q_1 \\ u_2(q_1, q_l|c_l) &= (48 - q_1 - q_l)q_l \\ u_2(q_1, q_h|c_h) &= (36 - q_1 - q_h)q_h \end{aligned}$$

2. Compute the Bayesian Equilibrium and the benefits of the firms in this equilibrium.

   Solution: *The best response of type* c_l *firm 2 is the solution of the following maximization problem*

   $$\max_{q_l} q_l(48 - q_1 - q_l)$$

   The solution is

   $$q_l = \frac{48 - q_1}{2} \tag{0.7}$$

The best response of type c_h firm 2 is the solution of the following maximization problem

$$\max_{q_h} q_h(36 - q_1 - q_h)$$

The solution is

$$q_h = \frac{36 - q_1}{2} \tag{0.8}$$

The best response of firm 1 is the solution of the following maximization problem

$$\max_{q_1} \frac{1}{4}(42 - q_1 - q_l)\, q_1 + \frac{3}{4}(42 - q_1 - q_h)\, q_1$$

The solution is

$$q_1 = \frac{168 - q_l - 3q_h}{8} \tag{0.9}$$

The Bayesian–Nash equilibrium is the solution to equations (0.7), (0.8) *and* (0.9). *We obtain,*

$$q_1^* = 15, \quad q_l^* = \frac{33}{2}, \quad q_h^* = \frac{21}{2}$$

The benefits are,

$$\Pi_1^* = 225 \quad \Pi_l^* = \frac{1089}{4} \quad \Pi_h^* = \frac{441}{4}$$

3. Suppose now that firm 1 knows that the costs of firm 2 is $c_2(q_2) = 12q_2$. Compute the Nash equilibrium and the benefits of the firms in this equilibrium.

   Solution: *The best response of type c_l firm 2 is the solution of the following maximization problem*

$$\max_{q_2} q_2(48 - q_1 - q_2)$$

The solution is

$$q_2 = \frac{48 - q_1}{2} \tag{0.10}$$

The best response of firm 1 is the solution of the following maximization problem

$$\max_{q_1} (42 - q_1 - q_2)\, q_1$$

The solution is

$$q_1 = \frac{42 - q_2}{2} \tag{0.11}$$

The Bayesian–Nash equilibrium is the solution to equations (0.10) *and* (0.11). *We obtain,*

$$\bar{q}_1 = 12, \quad \bar{q}_l = 18$$

The benefits are,

$$\bar{\Pi}_1 = 144 \quad \bar{\Pi}_l = 324$$

4. Suppose now that firm 1 knows that the costs of firm 2 is $c_2(q_2) = 24q_2$. Compute the Nash equilibrium and the benefits of the firms in this equilibrium.

   Solution: *The best response of type c_h firm 2 is the solution of the following maximization problem*

$$\max_{q_2} q_2(36 - q_1 - q_2)$$

The solution is

$$q_2 = \frac{36 - q_1}{2} \tag{0.12}$$

The best response of firm 1 is the same as in (0.10) .*The Bayesian–Nash equilibrium is the solution to equations* (0.10) *and* (0.12). *We obtain,*

$$\tilde{q}_1 = 16, \quad \tilde{q}_2 = 10$$

The benefits are,

$$\tilde{\Pi}_1 = 256 \quad \tilde{\Pi}_h = 100$$

5. In view of the above computation there is one type of firm 2 which prefers complete information and one type that prefers the situation with incomplete information. Identify those types.

   Solution: *Type* c_l *of firm 2 produces* $q_l = \frac{33}{2}$ *and obtains a profit of* $\Pi_l = \frac{1089}{4}$ *with incomplete information and produces* $q_l = 18$ *and obtains a profit of* $\Pi_l = 324$ *with complete information. It prefers the situation with complete information. This firm would benefit if it could credibly inform firm 1 that the the cost of firm 2 is low* c_l.

   Type c_h *of firm 2 produces* $q_h = \frac{21}{2}$ *and obtains a profit of* $\Pi_h = \frac{441}{4}$ *with incomplete information and produces* $q_h = 10$ *and obtains a profit of* $\Pi_h = 100$ *with complete information. It prefers the situation with incomplete information. This firm prefers that firm 1 is either not informed or (even better) that firm 1 believes (wrongly) that the the cost of firm 2 is low* c_l.

   Finally, firm 1 prefers the situation with incomplete information when it faces firm 2 with cost c_l. *And it prefers the situation with complete information when it faces firm 2 with cost* c_h.

Complete information c_l		**Incomplete information**		**Complete information** c_h
$\bar{q}_1 = 12$ $\bar{\Pi}_1 = 144$	$<$ $<$	$q_1^* = 15$ $\Pi_1^* = 225$	$<$ $<$	$\tilde{q}_1 = 16$ $\tilde{\Pi}_1 = 256$
$\bar{q}_l = 18$ $\bar{\Pi}_l = 324$	$>$ $>$	$q_l^* = 33/2$ $\Pi_l^* = 1089/4$		
		$q_h^* = 21/2$ $\Pi_h^* = 441/4$	$>$ $>$	$\tilde{q}_h = 10$ $\tilde{\Pi}_h = 100$

Problem 3: Consider a Cournot duopoly which operates in a market with inverse demand function $P(q) = a - q$, where $q = q_1 + q_2$ is total output in the market. Firm 2 knows if the value of total demand a is high ($a = a_h = 27$) or low ($a = a_l = 9$). The value of the parameter a is uncertain for firm 1. Firm 1 believes that with probability 2/3 it could be high ($a = a_h = 27$) and with probability 1/3 it could be low ($a = a_l = 9$). All the above is common knowledge and both firms choose simultaneously their production plans. Both firms have zero cost.

1. Represent this situation as a bayesian game.

   Solution: *There are two players,* $N = \{1, 2\}$. *Their types are* $T_1 = \{r\}$, $T_2 = \{a_l, a_h\}$ *where* a_l *represents the situation when firm 2 knows that* $a = a_l = 9$ *and* a_h *represents the situation when firm 2 knows that total demand is* $a = a_h = 27$. *The sets of strategies are* $S_1 = [0, \infty)$ *and* $S_2 = [0, \infty) \times [0, \infty) = \{(q_l, q_h) : q_l, q_h \in [0, \infty)\}$. *Here* q_l *(resp.* q_h*) represents the strategy of firm 2 when it knows that the demand is* $a_l = 9$ *(resp.* $a_h = 27$*). The beliefs are as follows,*

$$p_1(a_l|r) = \frac{1}{3}, \quad p_1(a_h|r) = \frac{2}{3}, \quad p_2(r|a_l) = p_2(r|a_h) = 1$$

   The utilities of the players are the following

$$\begin{aligned} u_1(q_1, q_2|a) &= \frac{1}{3}(9 - q_1 - q_l)q_1 + \frac{2}{3}(27 - q_1 - q_h)q_1 \\ u_2(q_1, q_2|a_l) &= (9 - q_1 - q_l)q_l \\ u_2(q_1, q_2|a_h) &= (27 - q_1 - q_h)q_h \end{aligned}$$

2. Compute the bayesian equilibrium and the profits in equilibrium.

   Solution: *The best response of type a_l firm 2 is the solution of the following maximization problem*

   $$\max_{q_l} q_l(9 - q_1 - q_l)$$

   The solution is

   $$q_l = \frac{9 - q_1}{2} \tag{0.13}$$

   The best response of type a_h firm 2 is the solution of the following maximization problem

   $$\max_{q_h} q_h(27 - q_1 - q_h)$$

   The solution is

   $$q_h = \frac{27 - q_1}{2} \tag{0.14}$$

   The best response of firm 1 is the solution of the following maximization problem

   $$\max_{q_1} \frac{1}{3}(9 - q_1 - q_l)\, q_1 + \frac{2}{3}(27 - q_1 - q_h)\, q_1$$

   The solution is

   $$q_1 = \frac{63 - q_l - 2q_h}{6} \tag{0.15}$$

   The Bayesian–Nash equilibrium is the solution to equations (0.13), (0.14) *and* (0.15). *We obtain,*

   $$q_1^* = 7, \quad q_l^* = 1, \quad q_h^* = 10$$

   The benefits are,

   $$\Pi_1^* = 49 \quad \Pi_l^* = 1 \quad \Pi_h^* = 100$$

3. Compare the above result with the one in which firm 1 knows the value of a.

   Solution: *Suppose first that firm 1 knows that the demand is $a = a_l = 9$. Then, the best response of firm 2 is given by* (0.13). *The best response of firm 1 is the solution of the following maximization problem*

   $$\max_{q_1} (9 - q_1 - q_2)\, q_1$$

   The solution is

   $$q_1 = \frac{9 - q_2}{2} \tag{0.16}$$

   The Bayesian–Nash equilibrium is the solution to equations (0.13) *and* (0.16). *We obtain,*

   $$\bar{q}_1 = \bar{q}_l = 3$$

   The benefits are,

   $$\bar{\Pi}_1 = \bar{\Pi}_l = 9$$

   Suppose now first that firm 1 knows that the demand is $a = a_h = 27$. Then, the best response of firm 2 is given by (0.14). *The best response of firm 1 is the solution of the following maximization problem*

   $$\max_{q_1} (27 - q_1 - q_2)\, q_1$$

   The solution is

   $$q_1 = \frac{27 - q_2}{2} \tag{0.17}$$

   The Bayesian–Nash equilibrium is the solution to equations (0.13) *and* (0.17). *We obtain,*

   $$\tilde{q}_1 = \tilde{q}_h = 9$$

   The benefits are,

   $$\tilde{\Pi}_1 = \tilde{\Pi}_h = 81$$

Complete information c_l		**Incomplete information**		**Complete information** c_h
$\bar{q}_1 = 3$	$<$	$q_1^* = 7$	$<$	$\tilde{q}_1 = 9$
$\bar{\Pi}_1 = 9$	$<$	$\Pi_1^* = 49$	$<$	$\tilde{\Pi}_1 = 81$
$\bar{q}_l = 30$	$>$	$q_l^* = 3$		
$\bar{\Pi}_l = 900$	$>$	$\Pi_l^* = 9$		
		$q_h^* = 10$	$>$	$\tilde{q}_h = 9$
		$\Pi_h^* = 100$	$>$	$\tilde{\Pi}_h = 81$

Problem 4: (The battle of the sexes) A couple is deciding wether to go to the soccer match or to the ballet. Each of the partners has to make the decision simultaneously and independently (they cannot communicate). She (player 1) likes soccer very much, but would prefer to go with her partner. He (player 2) enjoys ballet way more than soccer. Sometimes he prefers to go with his partner, but some other times he prefers to go alone (you can imagine the reasons). He knows his mood tonight. But, she does not. She thinks that the probability that he would enjoy her company tonight is 1/2. The situation is summarized in the following tables.

		He	
		S	B
She	S	4,2	0,0
	B	0, 0	2,4

He prefers her company

		He	
		S	B
She	S	4,0	0,4
	B	0, 2	2,0

He prefers to be alone

(a) Describe the situation as a Bayesian game.

Solution: *There are two players, $N = \{1, 2\}$. Their types are $T_1 = \{c\}$, $T_2 = \{a, b\}$ where a represents the situation when agent 2 knows that the payoffs are those in table A and b represents the situation when when agent 2 knows that the payoffs are those in table B. The sets of strategies are $S_1 = \{S, B\}$, $S_2 = \{SS, SB, BS, BB\}$. The beliefs of the players are*

$$p_1(a|c) = p_1(b|c) = \frac{1}{2}, \quad p_2(c|a) = p_2(c|b) = 1$$

The utilities of the players are given by the above tables. For example $u_1(B, B|c, a) = 2$, $u_2(S, B|c, b) = 4$.

(b) Find the Bayesian equililibria.

Solution: *Note that* $\text{BR}_2(S) = SB$, $\text{BR}_2(B) = BS$. *And*

$$\begin{aligned} u_1(S, SB) &= \frac{1}{2} \times 4 + \frac{1}{2} \times 0 = 2, \quad u_1(B, SB) = \frac{1}{2} \times 0 + \frac{1}{2} \times 2 = 1 \\ u_1(S, BS) &= \frac{1}{2} \times 0 + \frac{1}{2} \times 4 = 2, \quad u_1(B, BS) = \frac{1}{2} \times 2 + \frac{1}{2} \times 0 = 1 \end{aligned}$$

Hence, $\text{BR}_1(SB) = S$, $\text{BR}_1(BS) = S$. *Therefore, The NE in pure strategies is (S, SB).*

We compute now the BNE in mixed strategies. Suppose player 1 uses the mixed strategy $xS+(1-x)B$, player 2a uses the mixed strategy $yS + (1-y)B$ and player 2b uses the mixed strategy $zS + (1-z)B$. Note that

			P2	
			y	$1-y$
			S	B
P1	x	S	4,2	0,0
	1-x	B	0, 0	2,4

a

			P2	
			z	$1-z$
			S	B
P1	x	S	4,0	0,4
	1-x	B	0, 2	2,0

b

$$u_2(xS+(1-x)B, S|a) = 2x, \quad u_2(xS+(1-x)B, B|a) = 4(1-x)$$

Hence, player 2a is indifferent between the strategies S *and* B *if and only if* $2x = 4(1-x)$*, that is if and only if* $x = \frac{2}{3}$*. Now given this values of* x*,*

$$\begin{aligned} u_2(xS+(1-x)B, S|b)|_{x=\frac{2}{3}} &= 2(1-x)|_{x=\frac{2}{3}} = \frac{2}{3} \\ u_2(xS+(1-x)B, B|b)|_{x=\frac{2}{3}} &= 4x|_{x=\frac{2}{3}} = \frac{8}{3} \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{BR}_2(\frac{2}{3}S+\frac{1}{3}B|a) &= \{B, S\} \\ \mathrm{BR}_2(\frac{2}{3}S+\frac{1}{3}B|b) &= B \end{aligned}$$

In other words,

$$\mathrm{BR}_2(\frac{2}{3}S+\frac{1}{3}B) = \{BB, SB\}$$

That is, if player 1 follows the strategy $\frac{2}{3}S+\frac{1}{3}B$ *player 2a is indifferent between* S *and* B *and player 2b best response is* B*. Thus, we look for a BNE of the form*

$$\left(\frac{2}{3}S+\frac{1}{3}B, (yS+(1-y)B, B)\right)$$

If player 2 follows that above strategy, the expected payoffs of player 1 are

$$\begin{aligned} u_1(S, (yS+(1-y)B, B)) &= \frac{1}{2}(4y+0(1-y)) + \frac{1}{2}\times 0 = 2y \\ u_1(B, (yS+(1-y)B, B)) &= \frac{1}{2}(0y+2(1-y)) + \frac{1}{2}\times 2 = 2-y \end{aligned}$$

Hence, player 1 is indifferent between strategies S *and* B *if and only if* $2y = 2-y$*, that is if and only if* $y = \frac{2}{3}$*. It is now easy to check that the strategy*

$$\left(\frac{2}{3}S+\frac{1}{3}B, \left(\frac{2}{3}S+\frac{1}{3}B, B\right)\right)$$

is BNE. The (expected) payoffs of the players are

$$\begin{aligned} u_1\left(\frac{2}{3}S+\frac{1}{3}B, \left(\frac{2}{3}S+\frac{1}{3}B, B\right)\right) &= \frac{4}{3} \\ u_2\left(\frac{2}{3}S+\frac{1}{3}B, \left(\frac{2}{3}S+\frac{1}{3}B, B\right)\right) &= 2 \end{aligned}$$

We look now for another BNE in which player 2b uses mixed strategies. Note that

$$u_2(xS+(1-x)B, S|b) = 2(1-x), \quad u_2(xS+(1-x)B, B|b) = 4x$$

Hence, player 2b is indifferent between the strategies S *and* B *if and only if* $2(1-x) = 4x$*, that is if and only if* $x = \frac{1}{3}$*. Now given this values of* x*,*

$$\begin{aligned} u_2(xS+(1-x)B, S|a)|_{x=\frac{1}{3}} &= 2x|_{x=\frac{1}{3}} = \frac{2}{3} \\ u_2(xS+(1-x)B, B|a)|_{x=\frac{1}{3}} &= 4(1-x)|_{x=\frac{1}{3}} = \frac{8}{3} \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{BR}_2(\frac{1}{3}S+\frac{1}{3}B|a) &= B \\ \mathrm{BR}_2(\frac{1}{3}S+\frac{2}{3}B|b) &= \{B, S\} \end{aligned}$$

In other words,

$$\mathrm{BR}_2(\frac{1}{3}S + \frac{1}{3}) = \{BB, BS\}$$

That is, if player 1 follows the strategy $\frac{1}{3}S + \frac{2}{3}B$ *player 2b is indifferent between* S *and* B *and player 2a b's best response is* B*. Thus, we look for a BNE of the form*

$$\left(\frac{1}{3}S + \frac{2}{3}B, (B, zS + (1-z)B)\right)$$

If player 2 follows that above strategy, the expected payoffs of player 1 are

$$\begin{aligned}
u_1(S, (B, zS + (1-z)B)) &= \frac{1}{2}\left(4y + 0(1-y)\right) + \frac{1}{2} \times 0 = 2z \\
u_1(B, (B, zS + (1-z)B)) &= \frac{1}{2}\left(0y + 2(1-y)\right) + \frac{1}{2} \times 2 = 2 - z
\end{aligned}$$

Hence, player 1 is indifferent between strategies S *and* B *if and only if* $2z = 2 - z$*, that is if and only if* $z = \frac{2}{3}$*. It is now easy to check that the strategy*

$$\left(\frac{1}{3}S + \frac{2}{3}B, \left(B, \frac{2}{3}S + \frac{1}{3}B\right)\right)$$

is BNE. The (expected) payoffs of the players are

$$\begin{aligned}
u_1\left(\frac{1}{3}S + \frac{2}{3}B, \left(B, \frac{2}{3}S + \frac{1}{3}B\right)\right) &= \frac{4}{3} \\
u_2\left(\frac{1}{3}S + \frac{2}{3}B, \left(B, \frac{2}{3}S + \frac{1}{3}B\right)\right) &= 2
\end{aligned}$$

Solution 2: *Another way to find this is to note that,*

$$\begin{aligned}
u_1(S, SS) &= \frac{1}{2} \times 4 + \frac{1}{2} \times 4 = 4, \quad u_1(B, SS) = \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0 \\
u_1(S, SB) &= \frac{1}{2} \times 4 + \frac{1}{2} \times 0 = 2, \quad u_1(B, SB) = \frac{1}{2} \times 0 + \frac{1}{2} \times 2 = 1 \\
u_1(S, BS) &= \frac{1}{2} \times 0 + \frac{1}{2} \times 4 = 2, \quad u_1(B, BS) = \frac{1}{2} \times 2 + \frac{1}{2} \times 0 = 1 \\
u_1(S, BB) &= \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0, \quad u_1(B, BB) = \frac{1}{2} \times 2 + \frac{1}{2} \times 2 = 2
\end{aligned}$$

and

$$\begin{aligned}
u_2(S, SS) &= \frac{1}{2} \times 2 + \frac{1}{2} \times 0 = 1, \quad u_2(B, SS) = \frac{1}{2} \times 0 + \frac{1}{2} \times 2 = 1 \\
u_2(S, SB) &= \frac{1}{2} \times 2 + \frac{1}{2} \times 4 = 3, \quad u_2(B, SB) = \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0 \\
u_2(S, BS) &= \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0, \quad u_2(B, BS) = \frac{1}{2} \times 4 + \frac{1}{2} \times 2 = 3 \\
u_2(S, BB) &= \frac{1}{2} \times 0 + \frac{1}{2} \times 4 = 2, \quad u_2(B, BB) = \frac{1}{2} \times 4 + \frac{1}{2} \times 0 = 2
\end{aligned}$$

Now, we construct the table

		P2			
		SS	*SB*	*BS*	*BB*
P1	*S*	*4,1*	*2,3*	*2,0*	*0,2*
	B	*0,1*	*1,0*	*1,3*	*2,2*

Expected payoffs

Hence, the BNE in pure strategies is (S, SB).

We find now the BNE in mixed strategies. Suppose again that player 1 uses the mixed strategy $xS + (1-x)B$, *player 2a uses the mixed strategy* $yS + (1-y)B$ *and player 2b uses the mixed strategy* $zS + (1-z)B$. *Then*

$$p(SS) = yz, \quad p(SB) = y(1-z), \quad p(BS)(1-y)z, \quad p(BB) = (1-y)(1-z)$$

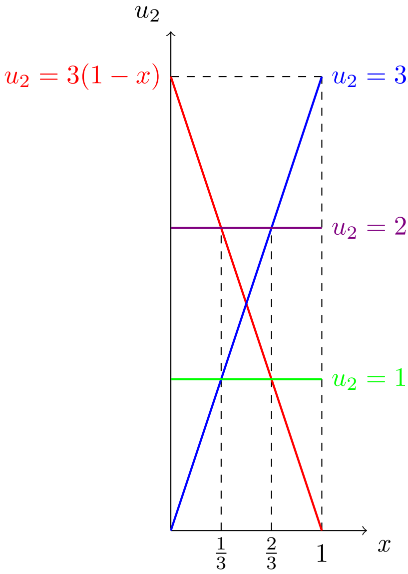
So, we get the table

P2

			yz	$y(1-z)$	$(1-y)z$	$(1-y)(1-z)$	
			SS	SB	BS	BB	
P1	*x*	*S*	*4,1*	*2,3*	*2,0*	*0,2*	*2 (y + z)*
	1-x	*B*	*0,1*	*1,0*	*1,3*	*2,2*	*2 - y - z*
			1	*3x*	*3(1-x)*	*2*	

Expected payoffs

Graphically,



We see that

$$\mathrm{BR}_2(x) = \begin{cases} BS & \textit{if} \quad 0 \le x < \frac{1}{3} \\ \{BS, BB\} & \textit{if} \quad x = \frac{1}{3} \\ BB & \textit{if} \quad \frac{1}{3} < x < \frac{2}{3} \\ \{SB, BB\} & \textit{if} \quad x = \frac{2}{3} \\ SB & \textit{if} \quad x > \frac{2}{3} \end{cases}$$

For $0 < x < \frac{1}{3}$, *the best reply of player 2 is* BS *and* $\mathrm{BR}_1(BS) = S$. *But,* $\mathrm{BR}_2(S) = SB$. *Hence, there is no BNE with* $0 < x < \frac{1}{3}$.

For $x = \frac{1}{3}$, *the best reply of player 2, type a is* B *and player 2, type b is indifferent between* S *and* B. *Thus, we must have* $y = 0$. *Player 1 follows a mixed strategy only if* $2y + 2z = 2 - y - z$. *Since,* $y = 0$ *this is equivalent to* $2z = 2 - z$, *that is* $z = \frac{2}{3}$. *We obtain the BNE*

$$\left(\frac{1}{3}S + \frac{2}{3}B, \left(B, \frac{2}{3}S + \frac{1}{3}B\right)\right).$$

For $\frac{1}{3} < x < \frac{2}{3}$, the best reply of player 2 is BB. Since,

$$u_1(S, BB) \;=\; \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0, \quad u_1(B, BB) = \frac{1}{2} \times 2 + \frac{1}{2} \times 2 = 2$$

we have that $\mathrm{BR}_1(BB) = B$. *But,* $\mathrm{BR}_2(B) = BS$. *Hence, there is no BNE with $\frac{1}{3} < x < \frac{2}{3}$.*

For $x = \frac{2}{3}$, the best reply of player 2, type b is B and player 2, type a is indifferent between S and B. Thus, we must have $z = 0$. Player 1 follows a mixed strategy only if $2y + 2z = 2 - y - z$. Since, $y = 0$ this is equivalent to $2y = 2 - y$, that is $y = \frac{2}{3}$. We obtain the BNE

$$\left(\frac{2}{3}S + \frac{1}{3}B, \left(\frac{2}{3}S + \frac{1}{3}B, B\right)\right)$$

Finally, for $\frac{2}{3} < x \leq 1$, the best reply of player 2 is SB. We have that $\mathrm{BR}_1(SB) = S$, *that is $x = 1$. Also,* $\mathrm{BR}_2(S) = SB$ *and we recover the BNE (S, SB).*

Problem 5: Consider the situation in which player 2 knows what game is played (A or B below). But player 1 only knows that A is played with probability p and B is played with probability $1 - p$.

Player 2

		S	B
Player 1	S	2,2	1,0
	B	1, 5	0,2

A

Player 2

		S	B
Player 1	S	2,2	0,5
	B	0, 0	4,2

B

(a) Describe the situation as a Bayesian game.

Solution: *There are two players, $N = \{1, 2\}$. Their types are $T_1 = \{c\}$, $T_2 = \{a, b\}$ where a represents the situation when agent 2 knows that the payoffs are those in table A and b represents the situation when when agent 2 knows that the payoffs are those in table B. The sets of strategies are $S_1 = \{S, B\}$, $S_2 = \{SS, SB, BS, BB\}$. The beliefs of the players are*

$$p_1\left(a|c\right) = p, \quad p\left(b|c\right) = 1 - p, \quad p_2\left(c|a\right) = p_2\left(c|b\right) = 1$$

The utilities of the players are given by the above tables. For example $u_1(B, B|c, a) = 0$, $u_2(S, B|c, b) = 5$.

(b) Find the Bayesian equilibria.

Solution: *Note that strategy B is dominated by strategy S for player 2a and strategy S is dominated by strategy B for player 2b. Hence, the BNE are of the form*

$$\left(xS + (1 - x)B, SB\right), \quad x \in [0, 1]$$

We remark now that

$$u_1(S, SB) = 2p, \quad u_1(B, SB) = 4 - 3p$$

Since $2p \geq 4 - 3p$ if and only if $p \geq \frac{4}{5}$ we have that

$$\mathrm{BR}_1(SB) = \begin{cases} S, & \textit{if } p > \frac{4}{5} \\ \{S, B\}, & \textit{if } p = \frac{4}{5} \\ B, & \textit{if } p < \frac{4}{5} \end{cases}$$

Therefore the NE are

$$\begin{array}{rl} (S, SB), & \textit{if } p > \frac{4}{5} \\ (xS + (1 - x)B, SB) \textit{ with } x \in [0, 1], & \textit{if } p = \frac{4}{5} \\ (B, SB), & \textit{if } p < \frac{4}{5} \end{array}$$

Problem 6: Consider the situation in which player 1 knows what game is played (A or B below). But player 2 only knows that A is played with probability 1/3 and B is played with probability 2/3.

		Player 2	
		S	B
Player 1	S	1,1	0,0
	B	0, 0	0,0

A

		Player 2	
		S	B
Player 1	S	0,0	0,0
	B	0, 0	2,2

B

(a) Describe the situation as a Bayesian game.

Solution: *There are two players,* $N = \{1, 2\}$. *Their types are* $T_1 = \{a, b\}$, $T_2 = \{c\}$ *where* a *represents the situation when agent 1 knows that the payoffs are those in table A and* b *represents the situation when when agent 1 knows that the payoffs are those in table B. The sets of strategies are* $S_1 = \{SS, SB, BS, BB\}$, $S_2 = \{S, B\}$. *The beliefs of the players are*

$$p_1\left(c|a\right) = p_1\left(c|b\right) = 1, \quad p_2\left(a|c\right) = \frac{1}{3}, \quad p\left(b|c\right) = \frac{2}{3}$$

The utilities of the players are given by the above tables.

(b) Find the Bayesian equililibria.

Solution: *Note that,*

$$\begin{aligned}
u_1(SS, S) &= \frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}, \quad u_1(SS, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0 \\
u_1(SB, S) &= \frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}, \quad u_1(SB, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 2 = \frac{4}{3} \\
u_1(BS, S) &= \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0, \quad u_1(BS, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0 \\
u_1(BB, S) &= \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0, \quad u_1(BB, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 2 = \frac{4}{3}
\end{aligned}$$

and

$$\begin{aligned}
u_2(SS, S) &= \frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}, \quad u_2(SS, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0 \\
u_2(SB, S) &= \frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}, \quad u_2(SB, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 2 = \frac{4}{3} \\
u_2(BS, S) &= \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0, \quad u_2(BS, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0 \\
u_2(BB, S) &= \frac{1}{3} \times 0 + \frac{2}{3} \times 0 = 0, \quad u_2(BB, B) = \frac{1}{3} \times 0 + \frac{2}{3} \times 2 = \frac{4}{3}
\end{aligned}$$

Now, we construct the table

		P2	
		S	*B*
P1	*SS*	$\frac{1}{3},\frac{1}{3}$	*0,0*
	SB	$\frac{1}{3},\frac{1}{3}$	$\frac{4}{3},\frac{4}{3}$
	BS	*0,0*	*0,0*
	BB	*0,0*	$\frac{4}{3},\frac{4}{3}$

Expected payoffs

An we look for a NE of the form

$$\begin{aligned}
\sigma_1 &= xSS + ySB + zBS + (1 - x - y - z)BB \\
\sigma_2 &= qS + (1 - q)B
\end{aligned}$$

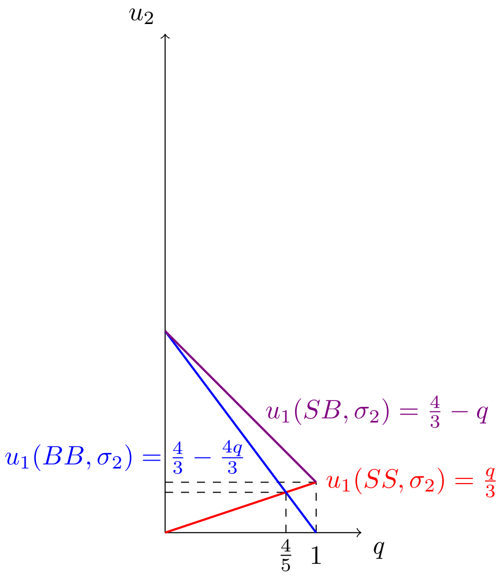
We get the table

P2

			q	*1 -q*	
			S	B	
	x	SS	$\frac{1}{3},\frac{1}{3}$	*0,0*	$\frac{q}{3}$
P1	y	SB	$\frac{1}{3},\frac{1}{3}$	$\frac{4}{3},\frac{4}{3}$	$\frac{4}{3}-q$
	z	BS	*0,0*	*0,0*	*0*
	1-x-y-z	BB	*0,0*	$\frac{4}{3},\frac{4}{3}$	$\frac{4}{3}-\frac{4q}{3}$
			$\frac{x+y}{3}$	$\frac{4}{3}(1-x-z)$	

Expected payoffs

Graphically,



We see that

$$\mathrm{BR}_1(q)=\begin{cases}\{SB,BB\} & \textit{if } q=0\\ \{SB\} & \textit{if } 0<q<1\\ \{SS,SB\} & \textit{if } q=1\end{cases}$$

We may assume $z=0$. *We obtain, the table*

P2

			q	*1 -q*	
			S	B	
	x	SS	$\frac{1}{3},\frac{1}{3}$	*0,0*	$\frac{q}{3}$
P1	y	SB	$\frac{1}{3},\frac{1}{3}$	$\frac{4}{3},\frac{4}{3}$	$\frac{4}{3}-q$
	1-x-y	BB	*0,0*	$\frac{4}{3},\frac{4}{3}$	$\frac{4}{3}-\frac{4q}{3}$
			$\frac{x+y}{3}$	$\frac{4}{3}(1-x)$	

Expected payoffs

- *Is there a BNE with* $q=0$*? In this equilibrium we must have that* $x=0$. *We obtain, the table*

P2

			$q=0$	$1-q=1$	
			S	B	
P1	y	SB	$\frac{1}{3},\frac{1}{3}$	$\frac{4}{3},\frac{4}{3}$	$\frac{4}{3}$
	1-y	BB	*0,0*	$\frac{4}{3},\frac{4}{3}$	$\frac{4}{3}$
			$\frac{y}{3}$	$\frac{4}{3}$	

Expected payoffs

We obtain the BNE

$$(ySB + (1-y)BB, B) \quad 0 \leq y \leq 1$$

with payoffs $u_1 = u_2 = \frac{4}{3}$.

- *Is there a BNE with* $0 < q < 1$*? In this equilibrium we must have that* $y = 1$*. But, then* $\mathrm{BR}_2(SB) = B$*. That is, player 2 best reply to* SB *is to choose* $q = 0$*. Hence, there is no BNE with* $0 < q < 1$.
- *Is there a BNE with* $q = 1$*? In this equilibrium we must have that* $x + y = 1$*. We obtain, the table*

			P2		
			$q = 1$	$1 - q = 0$	
			S	B	
P1	x	SS	$\frac{1}{3}, \frac{1}{3}$	*0,0*	$\frac{1}{3}$
	1-x	SB	$\frac{1}{3}, \frac{1}{3}$	$\frac{4}{3}, \frac{4}{3}$	$\frac{1}{3}$
			$\frac{1}{3}$	$\frac{4}{3}(1-x)$	

Expected payoffs

We need that

$$\frac{1}{3} \geq \frac{4}{3}(1-x)$$

that is $x \geq \frac{3}{4}$*. We obtain the following BNE*

$$(xSS + (1-x)SB, S) \quad \frac{3}{4} \leq x \leq 1$$

with payoffs $u_1 = u_2 = \frac{1}{3}$.

Problem 7: Two individuals consider donating towards a public good. If any of the agents contributes to the public good, then it is implemented. If agent agent $i = 1, 2$ contributes to the public good his utility is $u_i = 2 - c_i$. If agent $i = 1, 2$ does not contribute to the public good, but agent $j \neq i$ contributes to the public, the utility of agent i is $u_i = 2$. It is known that $c_1 = 1$. Only agent 2 knows c_2. Agent 1 knows that $c_2 = 1$ with probability p and $c_2 = 3$ with probability $1 - p$. That is player 2 knows what game is played (A or B below). But player 1 only knows that A is played with probability p and B is played with probability $1 - p$.

		Player 2	
		C	N
Player 1	C	1,1	1,2
	N	2, 1	0,0

A

		Player 2	
		C	N
Player 1	C	1,-1	1,2
	N	$2, -1$	0,0

B

(a) Describe the situation as a Bayesian game.

Solution: *There are two players,* $N = \{1, 2\}$*. Their types are* $T_1 = \{c\}$*,* $T_2 = \{a, b\}$ *where* a *represents the situation when agent 2 knows that the payoffs are those in table* A *and* b *represents the situation when when agent 2 knows that the payoffs are those in table* B*. The sets of strategies are* $S_1 = \{C, N\}$*,* $S_2 = \{CC, CN, NC, NN\}$*. The beliefs of the players are*

$$p_1(a|c) = p, \quad p(b|c) = 1 - p, \quad p_2(c|a) = p_2(c|b) = 1$$

The utilities of the players are given by the above tables. For example $u_1(C, N|c, a) = 1$*,* $u_2(N, C|c, b) = -1$.

(b) Find the Bayesian equililibria.

Solution: *Note that strategy C is dominated by strategy N for player 2b. Hence, the BNE are of the form $(*, *N)$. We remark that* $\mathrm{BR}_2(C|a) = N$, $\mathrm{BR}_2(N|a) = C$. *So,* $\mathrm{BR}_2(C) = NN$, $\mathrm{BR}_2(N) = CN$. *On the other hand,*

$$u_1(C, NN) = 1, \quad u_1(N, NN) = 0$$

So, $\mathrm{BR}_1(NN) = C$. *Hence, (C, NN) is a BNE, for any $0 \leq p \leq 1$. Note also that*

$$u_1(C, CN) = 1, \quad u_1(N, (CN) = 2p$$

$$\mathrm{BR}_1(CN) = \begin{cases} C & \textit{if } p < 1/2 \\ \{C, N\} & \textit{if } p = 1/2 \\ N & \textit{if } p > 1/2 \end{cases}$$

and we see that if $p \geq 1/2$, then (N, CN) is a BNE with payoffs $u_1 = 1$, $u_2 = 2$. We look now for a mixed strategy BNE. Suppose player 1 uses the mixed strategy $\sigma_1 = xC + (1-x)N$ and player 2a uses the mixed strategy $\sigma_2 = yC + (1-y)N$. Then, the expected payoffs are,

$$\begin{aligned} u_1(C, \sigma_2) &= 1 \\ u_1(N, \sigma_2) &= 2yp \end{aligned}$$

and

$$\begin{aligned} u_2(\sigma_1, CN|a) &= 1 \\ u_2(\sigma_1, NN|a) &= 2x \end{aligned}$$

So, $x = \frac{1}{2}$. Hence, for $p \geq \frac{1}{2}$, there is a mixed strategy BNE

$$\left(\frac{1}{2}C + \frac{1}{2}N, \left(\frac{1}{2p}C + \frac{2p-1}{2p}N, N\right)\right)$$

with payoffs $u_1 = u_2 = 1$.

Problem 8: Two individuals consider donating towards a public good. If any of the agents contributes to the public good, then it is implemented. If agent agent $i = 1, 2$ contributes to the public good his utility is $u_i = 2 - c_i$. If agent $i = 1, 2$ does not contribute to the public good, but agent $j \neq i$ contributes to the public, the utility of agent i is $u_i = 2$. That is

		Player 2	
		C	N
Player 1	C	$2 - c_1$, $2 - c_2$	$2 - c_1$,2
	N	2, $2 - c_2$	0,0

where c_1 and c_2 are distributed uniformly on the interval $[1, 3]$.

(a) Describe the situation as a Bayesian game.

Solution: *There are two players, $N = \{1, 2\}$. Their types are $T_1 = T_2 = [1, 3]$. The sets of strategies are $S_1 = S_2 = \{C, N\}$. The beliefs of the players are*

$$p_1(c_2 \leq c | c_1) = p_2(c_1 \leq c | c_2) = \begin{cases} 0 & \textit{if } c < 1 \\ \frac{c-1}{2} & \textit{if } c \in [1, 3] \\ 1 & \textit{if } c > 3 \end{cases},$$

The utilities of the players are given by the above tables.

(b) Show that there is a Bayesian equilibrium of the form $(s_1(c_1), s_2(c_2))$ with

$$s_i(c_i) = \begin{cases} C & \text{if } c_i \leq a \\ N & \text{if } c_i > a \end{cases}, \quad i = 1, 2$$

for some $a \in [1, 3]$: **Solution:** *The expected utility of agent 1 with the proposed strategy is*

$$\begin{aligned} u_1(C, s_2(c_2)) &= (2 - c_1)p\,(c_2 \leq a) + (2 - c_1)p\,(c_2 > a) = 2 - c_1 \\ u_1(N, s_2(c_2)) &= 2p\,(c_2 \leq a) + 0 \times p\,(c_2 > a) = \frac{2(a-1)}{2} = a - 1 \end{aligned}$$

Let $\varepsilon \geq 0$ *and let* $c_1 = a - \varepsilon$. *For agent 1, strategy* C *is a best reply iff* $2 - c_1 \geq a - 1$, *that is iff* $2 - a + \varepsilon \geq a - 1$. *Hence, we need that*

$$a \leq \frac{3}{2} + \frac{\varepsilon}{2} \quad \textit{for any } \varepsilon \geq 0$$

Now let $c_1 = a + \varepsilon$. *For agent 1 strategy* D *is a best reply iff* $2 - c_1 \leq a - 1$, *that is iff* $2 - a - \varepsilon \leq a - 1$. *Hence, we need that*

$$\frac{3}{2} - \frac{\varepsilon}{2} \leq a \quad \textit{for any } \varepsilon \geq 0$$

We see that if $a = \frac{3}{2}$, *then the strategy*

$$s_1(c_1) = \begin{cases} C & \textit{if } c_i \leq \frac{3}{2} \\ N & \textit{if } c_i > \frac{3}{2} \end{cases},$$

is a best reply to $s_2(c_2)$. *A similar argument shows that* $s_2(c_2)$ *is a best reply for agent 2 to* $s_1(c_1)$. *The payoffs in this equilibrium are*

$$u_i(s_1(c_1), s_2(c_2); c_i) = \begin{cases} 2 - c_i & \textit{if } c_i \leq \frac{3}{2} \\ \frac{1}{2} & \textit{if } c_i > \frac{3}{2} \end{cases}, \quad i = 1, 2$$

Problem 9:

Two risk averse individuals with utility functions $u(x) = \sqrt{x}$, where x represents money, face a first price auction. Agent $i = 1, 2$ $(i = 1, 2)$ values the good in v_i monetary units. This valuation is private information, but it is known that the v_i's are random variables independently and uniformly distributed in the interval $[0, 1]$.

(a) Describe the situation as a Bayesian game.

Solution: *There are two players,* $N = \{1, 2\}$. *Their types are* $T_1 = T_2 = [0, 1]$. *The sets of strategies are* $S_i(v_i) = [0, v_i]$ $i = 1, 2$. *The beliefs of the players are*

$$p_1\,(v_2 \leq c | v_1) = p_2\,(v_1 \leq c | v_2) = \begin{cases} o & \textit{if } c < 0 \\ c & \textit{if } c \in [0, 1] \\ 1 & \textit{if } c\ > 1 \end{cases},$$

The utilities of the players are

$$u_i(b_1, b_2; v_i) = \begin{cases} 0 & \textit{if } b_i < b_j \\ \frac{\sqrt{v_i - b_i}}{2} & \textit{if } b_i = b_j \\ \sqrt{v_i - b_i} & \textit{if } b_i > b_j \end{cases} \quad i = 1, 2, \quad i \neq j$$

(b) Find a bayesian Nash equilibrium of the form $b_i(v_i) = \alpha_i v_i$. What is the utility of each individual in equilibrium?

Solution: *Suppose player 2 follows the strategy* $b_2(v_2) = \alpha_2 v_2$. *If player 1 chooses to bid* b_1, *his expected utility is*

$$\begin{aligned} u_1(b_1|v_1) &= p(b_1 > b_2(v_2))\sqrt{v_1 - b_1} + p(b_1 = b_2(v_2))\frac{1}{2}\sqrt{v_1 - b_1} + 0 \times p(b_1\,(v_1) < b_2(v_2)) \\ &= \sqrt{v_1 - b_1} p(b_1 > b_2(v_2)) \end{aligned}$$

because $p(b_1 = b_2(v_2)) = 0$. *Thus,*

$$u_1(b_1|v_1) = p(b_1 > b_2(v_2))\sqrt{v_1 - b_1} = p(b_1 > \alpha_2 v_2)\sqrt{v_1 - b_1} = p\left(v_2 < \frac{b_1}{\alpha_2}\right)\sqrt{v_1 - b_1} = \frac{b_1\sqrt{v_1 - b_1}}{\alpha_2}$$

The best reply of player 1 is given my the solution to

$$\max_{b_1} b_1\sqrt{v_1 - b_1}$$

The first order condition is

$$\sqrt{v_1 - b_1} = \frac{b_1}{2\sqrt{v_1 - b_1}}$$

whose solution is

$$b_1(v_1) = \frac{2}{3}v_1$$

Similarly, if player 1 follows the strategy $b_1(v_1) = \alpha_1 v_1$ *the best reply for player 2 is*

$$b_2(v_2) = \frac{2}{3}v_2$$

Hence

$$b_i(v_i) = \frac{2}{3}v_i, \quad i = 1, 2$$

is BNE. The expected payoff of agent $i = 1, 2$ *is*

$$u_i\left(b_1(v_1), b_2(v_2); v_i\right) = \frac{\frac{2}{3}v_i\sqrt{v_i - \frac{2}{3}v_i}}{\frac{2}{3}} = \sqrt{\frac{2}{3}}v_i^{\frac{3}{2}}$$