University Carlos III of Madrid **Departament of Economics**

Game Theory: Problem set 1. Solutions.

Exercise 1: For each one of the following normal form games find

(a) The pure-strategy Nash equilibria and their payoffs;

(b) The mixed-strategy Nash equilibria and their payoffs;

(c) Are there any equilibria in dominant strategies? If yes, which ones?

(a)

	C	D
A	$-50, -50$	$100, 0$
B	$0, 100$	$0, 0$

Solution: *The best responses of the players are*

	C	D
A	$-50, -50$	$\underline{100}, \underline{0}$
B	$\underline{0}, \underline{100}$	$0, 0$

There are two NE in pure strategies: (A, D) *and* (B, C)*. We look for a NE in mixed strategies of the form*

$$\begin{aligned} \sigma_1 &= xA + (1-x)B \\ \sigma_2 &= yC + (1-y)D \end{aligned}$$

The expected utilities of the players are

$$\begin{aligned} u_1(A, \sigma_2) &= -50y + 100(1-y) = 100 - 150y \\ u_1(B, \sigma_2) &= 0 \times y + 0 \times (1-y) = 0 \end{aligned}$$

and

$$\begin{aligned} u_2(\sigma_1, C) &= 50x + 100(1-x) = 100 - 150x \\ u_2(\sigma_1, D) &= 0 \times x + 0 \times (1-x) = 0 \end{aligned}$$

Equating the utilities of the players

$$100 - 150y = 0, \quad 100 - 150x = 0$$

we obtain $x = y = \frac{2}{3}$*. Thus, there is an additional NE in mixed strategies,*

$$\begin{aligned} \sigma_1 &= \frac{2}{3}A + \frac{1}{3}B \\ \sigma_2 &= \frac{2}{3}C + \frac{1}{3}D \end{aligned}$$

(b)

	C	D
A	$-1, 2$	$1, -2$
B	$1, -2$	$-1, 2$

Solution: *The best responses of the players are*

	C	D
A	$-1, \underline{2}$	$\underline{1}, -2$
B	$\underline{1}, -2$	$-1, \underline{2}$

There are no NE in pure strategies. We look for a NE in mixed strategies of the form

$$\sigma_1 = xA + (1-x)B$$
$$\sigma_2 = yC + (1-y)D$$

The expected utilities of the players are

$$u_1(A, \sigma_2) = -y + (1-y) = 1 - 2y$$
$$u_1(B, \sigma_2) = y - (1-y) = 2y - 1$$

and

$$u_2(\sigma_1, C) = 2x - 2(1-x) = 4x - 2$$
$$u_2(\sigma_1, D) = -2x + 2(1-x) = 2 - 4x$$

Equating the utilities of the players

$$1 - 2y = 2y - 1, \quad 4x - 2 = 2 - 4x$$

we obtain $x = y = \frac{1}{2}$. *Thus, there is an additional NE in mixed strategies,*

$$\sigma_1 = \frac{1}{2}A + \frac{1}{2}B$$
$$\sigma_2 = \frac{1}{2}C + \frac{1}{2}D$$

(c)

	C	D
A	$-10, -10$	$0, -20$
B	$-20, 0$	$10, 10$

Solution: *The best responses of the players are*

	C	D
A	$-\underline{10}, -\underline{10}$	$0, -20$
B	$-20, 0$	$\underline{10}, \underline{10}$

There are two NE in pure strategies: (A, C) *and* (B, D). *We look for a NE in mixed strategies of the form*

$$\sigma_1 = xA + (1-x)B$$
$$\sigma_2 = yC + (1-y)D$$

The expected utilities of the players are

$$u_1(A, \sigma_2) = -10y + 0 \times (1-y) = -10y$$
$$u_1(B, \sigma_2) = -20y + 10(1-y) = 10 - 30y$$

and

$$u_2(\sigma_1, C) = -10x + 0 \times (1-x) = -10x$$
$$u_2(\sigma_1, D) = -20x + 10(1-x) = 10 - 30x$$

Equating the utilities of the players

$$-10y = 10 - 30y, \quad -10x = 10 - 30x$$

we obtain $x = y = \frac{1}{2}$. *Thus, there is an additional NE in mixed strategies,*

$$\sigma_1 = \frac{1}{2}A + \frac{1}{2}B$$
$$\sigma_2 = \frac{1}{2}C + \frac{1}{2}D$$

(d)

	C	D
A	$3,3$	$0,0$
B	$0,0$	$1,1$

Solution: *We look for a NE of the form*

$$\sigma_1 = xA + (1-x)B$$
$$\sigma_2 = yC + (1-y)D$$

The expected utilities of the players are

$$u_1(\sigma_1, \sigma_2) = 1 - x - y + 4xy = 1 - y + x(4y-1)$$
$$u_2(\sigma_1, \sigma_2) = 1 - x - y + 4xy = 1 - x + y(4x-1)$$

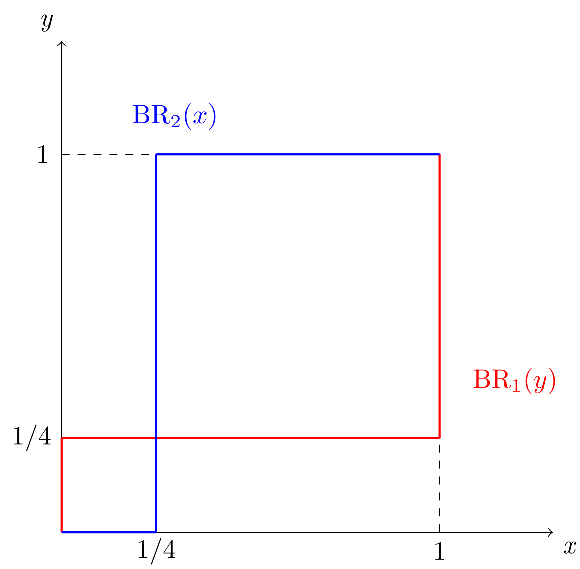
We see that

$$\mathrm{BR}_1(y) = \begin{cases} 1 & \text{if } y > \frac{1}{4} \\ [0,1] & \text{if } y = \frac{1}{4} \\ 0 & \text{if } y < \frac{1}{4} \end{cases}$$

and

$$\mathrm{BR}_2(x) = \begin{cases} 1 & \text{if } x > \frac{1}{4} \\ [0,1] & \text{if } x = \frac{1}{4} \\ 0 & \text{if } x < \frac{1}{4} \end{cases}$$

Graphically,



And we obtain three NE

(a) (A, C) *with payoffs* $u_1 = u_2 = 3$.

(b) (B, D) *with payoffs* $u_1 = u_2 = 1$.

(c) $\left(\frac{1}{4}A + \frac{3}{4}B, \frac{1}{4}C + \frac{3}{4}D\right)$ *with payoffs* $u_1 = u_2 = \frac{3}{4}$.

(e)

	C	D
A	$4,-8$	$-2,-7$
B	$1,-5$	$-1,-6$

Solution: *We look for a NE of the form*

$$\sigma_1 = xA + (1-x)B$$
$$\sigma_2 = yC + (1-y)D$$

The expected utilities of the players are

$$\begin{aligned} u_1(\sigma_1,\sigma_2) &= -1 - x + 2y + 4xy = -1 + 2y + x(4y-1) \\ u_2(\sigma_1,\sigma_2) &= -6 - x + y - 2xy = -6 - x + y(1-2x) \end{aligned}$$

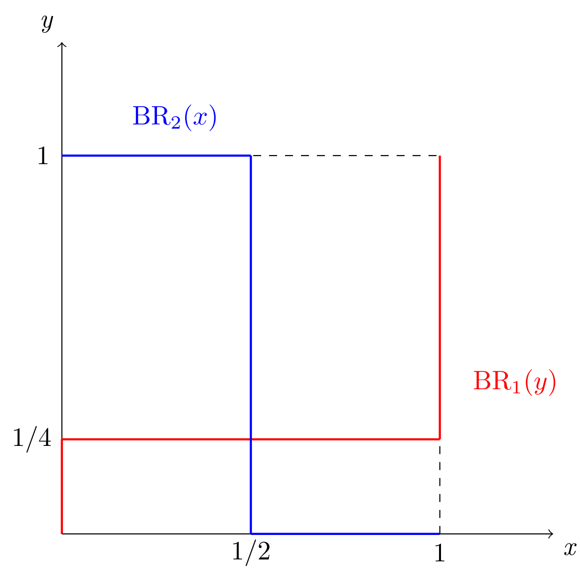
We see that

$$\mathrm{BR}_1(y) = \begin{cases} 1 & \text{if } y > \frac{1}{4} \\ [0,1] & \text{if } y = \frac{1}{4} \\ 0 & \text{if } y < \frac{1}{4} \end{cases}$$

and

$$\mathrm{BR}_2(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2} \\ [0,1] & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2} \end{cases}$$

Graphically,



And we obtain a unique NE

$$(A,C), \quad (B,D), \quad \left(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{4}C + \frac{3}{4}D\right)$$

with payoffs

$$u_1 = -\frac{1}{2}, \quad u_2 = -\frac{13}{2}$$

(f)

	C	D
A	$2,5$	$8,-1$
B	$4,-8$	$-2,-3$

Solution: *We look for a NE of the form*

$$\begin{aligned} \sigma_1 &= xA + (1-x)B \\ \sigma_2 &= yC + (1-y)D \end{aligned}$$

The expected utilities of the players are

$$\begin{aligned} u_1(\sigma_1,\sigma_2) &= -2 + 10x + 6y - 12xy = -2 + 6y + x(10-12y) \\ u_2(\sigma_1,\sigma_2) &= -3 + 2x - 5y + 11xy = -3 + 2x + y(11x-5) \end{aligned}$$

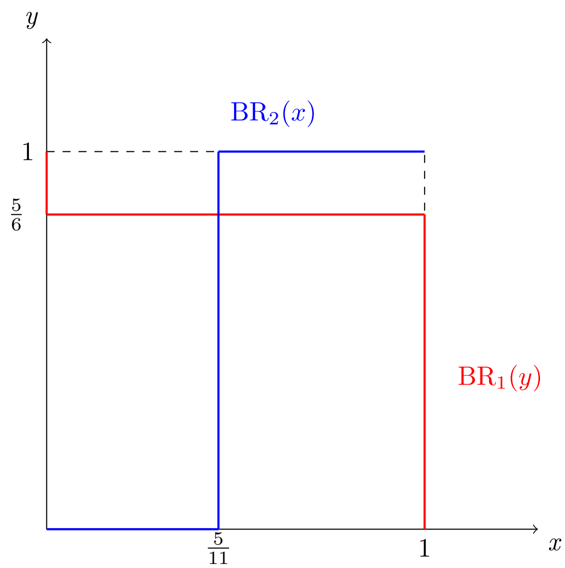
We see that

$$\mathrm{BR}_1(y) = \begin{cases} 1 & \textit{if } y < \frac{5}{6} \\ [0,1] & \textit{if } y = \frac{5}{6} \\ 0 & \textit{if } y > \frac{5}{6} \end{cases}$$

and

$$\mathrm{BR}_2(x) = \begin{cases} 1 & \textit{if } x > \frac{5}{11} \\ [0,1] & \textit{if } x = \frac{5}{11} \\ 0 & \textit{if } x < \frac{5}{1} \end{cases}$$

Graphically,



And we obtain a unique NE

$$(A, C), \quad (B, D), \quad \left(\frac{5}{11}A + \frac{6}{11}B, \frac{5}{6}C + \frac{1}{6}D\right)$$

with payoffs

$$u_1 = 3 \quad u_2 = -\frac{23}{11}$$

(g)

	C	D
A	$10, 10$	$-2, -7$
B	$1, -5$	$1, 1$

Solution: *We look for a NE of the form*

$$\begin{aligned} \sigma_1 &= xA + (1-x)B \\ \sigma_2 &= yC + (1-y)D \end{aligned}$$

The expected utilities of the players are

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= 1 - 3x + 12xy = 1 + x(12y - 3) \\ u_2(\sigma_1, \sigma_2) &= 1 - 8x - 6y + 23xy = 1 - 8x + y(23x - 6) \end{aligned}$$

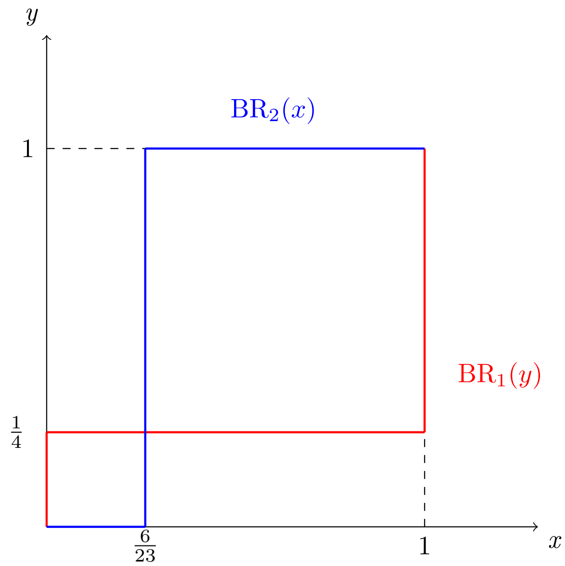
We see that

$$\mathrm{BR}_1(y) = \begin{cases} 0 & \textit{if } y < \frac{1}{4} \\ [0,1] & \textit{if } y = \frac{1}{4} \\ 1 & \textit{if } y > \frac{1}{4} \end{cases}$$

and

$$\mathrm{BR}_2(x) = \begin{cases} 1 & \textit{if } x > \frac{6}{23} \\ [0,1] & \textit{if } x = \frac{6}{23} \\ 0 & \textit{if } x < \frac{6}{23} \end{cases}$$

Graphically,



And we obtain three NE

(a) (A, C) *with payoffs* $u_1 = u_2 = 10$.

(b) (B, D) *with payoffs* $u_1 = u_2 = 1$.

(c) $\left(\frac{6}{23}A + \frac{17}{23}B, \frac{1}{4}C + \frac{3}{4}D\right)$ *with payoffs* $u_1 = 1$, $u_2 = -\frac{25}{23}$.

Exercise 2: For the normal form games below determine

(a) the strategies that survive the iterated elimination of strictly dominated strategies.

(b) the pure strategy Nash equilibria of the game.

(c) the mixed strategy Nash equilibria of the game.

(a)

	X	Y	Z
A	20, 10	10, 20	1, 1
B	10, 20	20, 10	1, 1
C	1, 1	1, 1	0, 0

Solution: *Note that* C *is dominated by* A *for player 1. Hence, we obtain the game*

	X	Y	Z
A	20, 10	10, 20	1, 1
B	10, 20	20, 10	1, 1

Now that Z *is dominated by* X *for player 2. Hence, we obtain the game*

	X	Y
A	20, 10	10, 20
B	10, 20	20, 10

The rationalizable strategies are $\{A, B\} \times \{X, Y\}$. *The best replies of the players are*

	X	Y
A	$\underline{20}, 10$	$10, \underline{20}$
B	$10, \underline{20}$	$\underline{20}, 10$

We see that there are no NE in pure strategies. Let us look for a NE in mixed strategies of the form

$$\sigma_1 = xA + (1-x)B$$
$$\sigma_2 = yX + (1-y)Y$$

The expected utilities of player 1 when he chooses pure actions are

$$u_1(A, \sigma_2) = 20y + 10(1-y) = 10 + 10y$$
$$u_1(B, \sigma_2) = 10y + 20(1-y) = 20 - 10y$$

We must have that $10 + 10y = 20 - 10y$*. That is* $y = \frac{1}{2}$*.*
The expected utilities of player 2 when he chooses pure actions are

$$u_2(\sigma_1, X) = 10x + 20(1-x) = 20 - 10x$$
$$u_2(\sigma_1, Y) = 20x + 10(1-x) = 10 - 20x$$

Equating $20 - 10x = 10 - 20x$*, we obtain* $x = \frac{1}{2}$*. We conclude that there is a NE in mixed strategies,*

$$\sigma_1 = \frac{1}{2}A + \frac{1}{2}B$$
$$\sigma_2 = \frac{1}{2}X + \frac{1}{2}Y$$

(b)

	D	E	F
A	$2, 0$	$1, 1$	$4, 2$
B	$3, 4$	$1, 2$	$2, 3$
C	$1, 3$	$0, 2$	$3, 0$

Solution: *Note that* C *is dominated by* A *for player 1,*

	D	E	F
A	$2, 0$	$1, 1$	$4, 2$
B	$3, 4$	$1, 2$	$2, 3$

And now E *is dominated by* F *for player 2,*

	D	F
A	$2, 0$	$4, 2$
B	$3, 4$	$2, 3$

The rationalizable strategies are $\{A, B\} \times \{D, F\}$*. The best responses are*

	D	F
A	$2, 0$	$\underline{4}, \underline{2}$
B	$\underline{3}, \underline{4}$	$2, 3$

and we obtain that the NE in pure strategies are (B, D) *and* (A, F)*.*

We look now for a NE in mixed strategies of the form

$$\sigma_1 = xA + (1-x)B$$
$$\sigma_2 = yD + (1-x)F$$

The expected utilities of player 1 when he chooses pure actions are

$$u_1(A, \sigma_2) = 2y + 4(1-y) = 4 - 2y$$
$$u_1(B, \sigma_2) = 3y + 2(1-y) = 2 + y$$

We must have that player 1 is indifferent between A, B. That is $4-2y=2+y$. *Therefore,* $y=2/3$. *The expected utilities of player 2 when he chooses pure actions are*

$$\begin{aligned} u_2(\sigma_1, D) &= 0x+4(1-x)=4-4x \\ u_2(\sigma_1, F) &= 2x+3(1-x)=3-x \end{aligned}$$

We must have that player 2 is indifferent between D and F. That is, $4-4x=3-x$. *Therefore,* $x=1/3$. *We obtain the NE*

$$\left(\frac{1}{3}A+\frac{2}{3}B, \frac{2}{3}D+\frac{1}{3}F\right)$$

with expected payoffs $u_1=u_2=\frac{8}{3}$.

(c)

	D	E	F
A	$2,2$	$1,1$	$4,3$
B	$1,4$	$0,5$	$3,3$
C	$3,3$	$-2,2$	$3,0$

Solution: *Note that B is dominated by A for player 1,*

	D	E	F
A	$2,2$	$1,1$	$4,3$
C	$3,3$	$-2,2$	$3,0$

And now E is dominated by D for player 2,

	D	F
A	$2,2$	$4,3$
C	$3,3$	$3,0$

The rationalizable strategies are $\{A, C\} \times \{D, F\}$. *The best responses are*

	D	F
A	$2,2$	$\underline{4},\underline{3}$
C	$\underline{3},\underline{3}$	$3,0$

and we obtain that the NE in pure strategies are (C, D) *and* (A, F).

We look now for a NE in mixed strategies of the form

$$\begin{aligned} \sigma_1 &= xA+(1-x)C \\ \sigma_2 &= yD+(1-x)F \end{aligned}$$

The expected utilities of player 1 when he chooses pure actions are

$$\begin{aligned} u_1(A, \sigma_2) &= 2y+4(1-y)=4-2y \\ u_1(C, \sigma_2) &= 3y+3(1-y)=3 \end{aligned}$$

We must have that player 1 is indifferent between A and C. That is $4-2y=3$. *Therefore,* $y=\frac{1}{2}$. *The expected utilities of player 2 when he chooses pure actions are*

$$\begin{aligned} u_2(\sigma_1, D) &= 2x+3(1-x)=3-x \\ u_2(\sigma_1, F) &= 3x+0(1-x)=3x \end{aligned}$$

We must have that player 2 is indifferent between D and F. That is, $3-x=3x$. *Therefore,* $x=\frac{3}{4}$. *We obtain the NE*

$$\left(\frac{3}{4}A+\frac{1}{4}B, \frac{1}{2}D+\frac{1}{2}F\right)$$

with expected payoffs $u_1=3$, $u_2=\frac{9}{4}$.

(d)

	D	E	F
A	$4,-1$	$3,4$	$2,0$
B	$1,4$	$-3,5$	$-3,-3$
C	$3,3$	$2,-2$	$8,-3$

Solution: *Note that B is dominated by A for player 1,*

	D	E	F
A	$4,-1$	$3,4$	$2,0$
C	$3,3$	$2,-2$	$8,-3$

and F is dominated by E for player 2,

	D	E
A	$4,-1$	$3,4$
C	$3,3$	$2,-2$

Now, C is dominated by A for player 1,

	D	E
A	$4,-1$	$3,4$

And D is dominated by E for player 2. The set of rationalizable strategies is $\{A\} \times \{E\}$. The unique NE is (A, E).

(e)

	D	E	F
A	$2,-1$	$2,-1$	$4,-2$
B	$2,-1$	$-2,-1$	$3,-2$
C	$1,3$	$-2,2$	$3,4$

Solution: *Note that C is dominated by A for player 1,*

	D	E	F
A	$2,-1$	$2,-1$	$4,-2$
B	$2,-1$	$-2,-1$	$3,-2$

and F is dominated by D for player 2,

	D	E
A	$2,-1$	$2,-1$
B	$2,-1$	$-2,-1$

The best responses of the players are

	D	E
A	$\underline{2},\underline{-1}$	$\underline{2},\underline{-1}$
B	$\underline{2},\underline{-1}$	$-2,\underline{-1}$

We have three NE in pure strategies: (A, D), (A, E) and (B, D). We look now for a NE in mixed strategies of the form

$$\begin{aligned} \sigma_1 &= xA + (1-x)B \\ \sigma_2 &= yD + (1-y)E \end{aligned}$$

The expected utilities of player 1 when he chooses pure actions are

$$\begin{aligned} u_1(A, \sigma_2) &= 2y + 2(1-y) = 2 \\ u_1(B, \sigma_2) &= 2y - 2(1-y) = -2 + 4y \end{aligned}$$

Now, for $0 \le y < 1$ we have that $u_1(A, \sigma_2) > -2 + 4y$. Hence, in a mixed NE, player 1 will not choose strategy B with positive probability unless $y = 1$. The expected utilities of player 2 when he chooses pure actions are

$$\begin{aligned} u_2(\sigma_1, D) &= -x - (1-x) = -1 \\ u_2(\sigma_1, E) &= -x + -(1-x) = -1 \end{aligned}$$

for any $0 \le x \le 1$. Player 2 is indifferent among all his strategies. We obtain infinitely many NE of the form,

$$\begin{aligned} \sigma_1 &= xA + (1-x)B, \quad 0 \le x \le 1 \\ \sigma_2 &= D \end{aligned}$$

with payoffs $u_1 = 2$, $u_2 = -1$. And we also obtain infinitely many NE of the form,

$$\begin{aligned} \sigma_1 &= A \\ \sigma_2 &= yD + (1-y)E, \quad 0 \le y \le 1 \end{aligned}$$

with payoffs $u_1 = 2$, $u_2 = -1$.

(f)

	A	B	C	D
X	$1,4$	$2,2$	$4,1$	$2,0$
Y	$0,0$	$1,3$	$2,5$	$0,0$
Z	$0,1$	$5,0$	$8,1$	$3,3$

Solution: *Note that Y is dominated by X for player 1,*

	A	B	C	D
X	$1,4$	$2,2$	$4,1$	$2,0$
Z	$0,1$	$5,0$	$8,1$	$3,3$

Now, B is dominated by A for player 2,

	A	C	D
X	$1,4$	$4,1$	$2,0$
Z	$0,1$	$8,1$	$3,3$

For the moment, we consider the strategies $S_1\{X, Y\}$, $S_2 = \{A, C, D\}$. The best responses of the player are

	A	C	D
X	$\underline{1},\underline{4}$	$4,1$	$2,0$
Z	$0,1$	$\underline{8},1$	$\underline{3},\underline{3}$

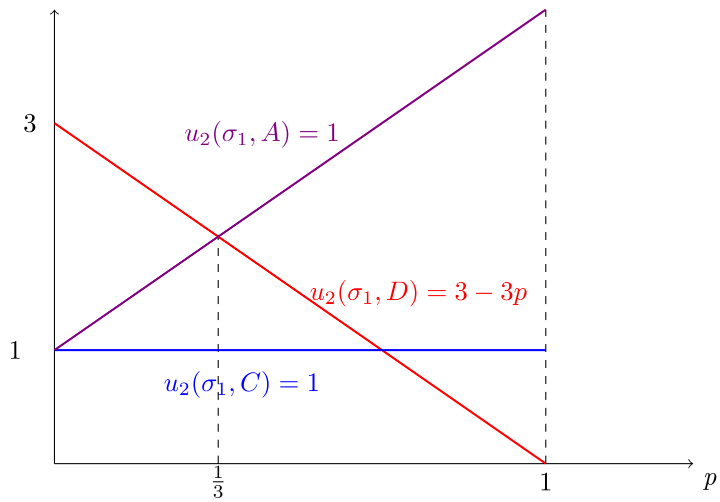
We obtain two pure strategy NE: (X, A) and (Z, D). We look now for a NE in mixed strategies of the form

$$\begin{aligned} \sigma_1 &= pX + (1-p)Z \\ \sigma_2 &= xA + yC + (1-x-y)D \end{aligned}$$

The expected utilities of player 2 when he chooses pure actions are

$$\begin{aligned} u_2(\sigma_1, A) &= 4p + (1-p) = 1 + 3p \\ u_2(\sigma_1, C) &= p + (1-p) = 1 \\ u_2(\sigma_1, D) &= 0p + 3(1-p) = 3 - 3p \end{aligned}$$

We represent this three functions graphically,



We see in this picture, that player 2 does not want to consider strategy C. In fact, consider now the mixed strategy

$$\sigma_2 = \frac{1}{2}A + \frac{1}{2}D$$

for player 2. Note that

$$\begin{aligned} u_2(X, \sigma_2) &= 2 > 1 = u_2(X, C) \\ u_2(Z, \sigma_2) &= 2 > 1 = u_2(Z, C) \end{aligned}$$

That is, strategy C is (strictly) dominated the the mixed strategy σ_2. We may eliminate strategy C. The rationalizable strategies are $\{X, Y\} \times \{A, D\}$. The best responses of the player are

	A	D
X	$\underline{1}, \underline{4}$	$2, 0$
Z	$0, 1$	$\underline{3}, \underline{3}$

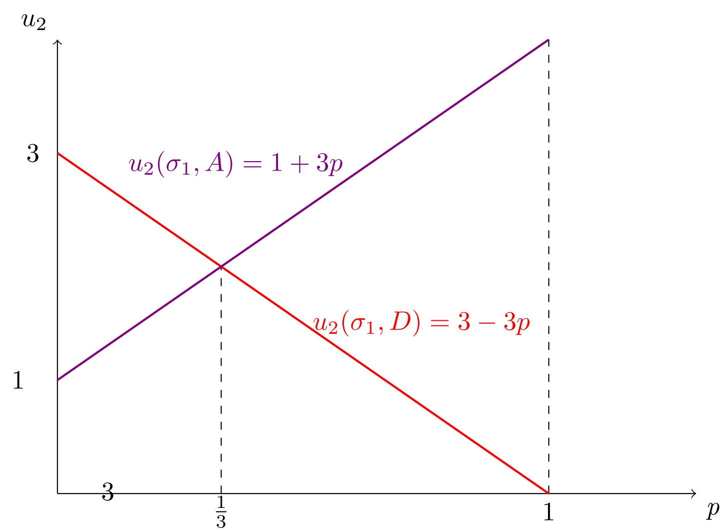
We obtain two pure strategy NE: (X, A) and (Z, D). We look now for a NE in mixed strategies of the form

$$\begin{aligned} \sigma_1 &= pX + (1-p)Z \\ \sigma_2 &= xA + (1-x)D \end{aligned}$$

The expected utilities of player 2 when he chooses pure actions are

$$\begin{aligned} u_2(\sigma_1, A) &= 4p + (1-p) = 1 + 3p \\ u_2(\sigma_1, D) &= 0p + 3(1-p) = 3 - 3p \end{aligned}$$

We represent this three functions graphically,



We see that

$$\text{BR}_2(\sigma_1) = \begin{cases} D & (x = y = 0) & \textit{if} \quad 0 \le p < \frac{1}{3} \\ \{A, D\} & (x \in [0,1] y = 0) & \textit{if} \quad p = \frac{1}{3} \\ A & (x = 1) & \textit{if} \quad \frac{1}{3} < p \le 1 \end{cases}$$

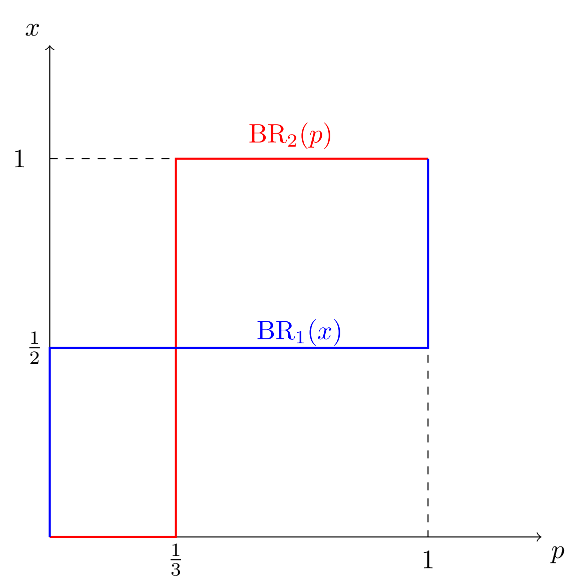
The expected utilities of player 1 when he chooses pure actions and $y = 0$ *are*

$$\begin{aligned} u_1(X, \sigma_2) &= x + 2(1 - x) = 2 - x \\ u_1(Z, \sigma_2) &= 0x + 3(1 - x) = 3 - 3x \end{aligned}$$

Hence,

$$\text{BR}_1(\sigma_2) = \begin{cases} Z & (p = 0) & \textit{if} \quad 0 \le x < \frac{1}{2} \\ \{X, Z\} & (p \in [0,1]) & \textit{if} \quad x = \frac{1}{2} \\ X & (p = 1) & \textit{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

Graphically,



And we obtain three NE

(a) $x = y = 0$; $p = 0$. *That is,* (Z, D), *with payoffs* $u_1 = u_2 = 3$.

(b) $x = 1$, $y = 0$; $p = 1$. *That is,* (A, X), *with payoffs* $u_1 = 1$, $u_2 = 4$.

(c) $x = \frac{1}{2}$, $y = 0$; $p = \frac{1}{3}$. *That is,*

$$\begin{aligned} \sigma_1 &= \frac{1}{3}X + \frac{2}{3}Z \\ \sigma_2 &= \frac{1}{2}A + \frac{1}{2}D \end{aligned}$$

with payoffs $u_1 = \frac{3}{2}$, $u_2 = 2$.

Exercise 3: For the normal form game

	X	Y	Z
A	$10, 30$	$0, 20$	$20, 30$
B	$15, 35$	$10, 40$	$10, 40$
C	$25, 25$	$5, 25$	$5, 25$

determine all the Nash equilibria of the game.

Solution: *There are no (strictly) dominated strategies. The best replies of the players are*

	X	Y	Z
A	$10, \underline{30}$	$0, 20$	$\underline{20}, \underline{30}$
B	$15, 35$	$\underline{10}, \underline{40}$	$10, \underline{40}$
C	$\underline{25}, \underline{25}$	$5, \underline{25}$	$5, 25$

We obtain three NE in pure strategies, (A, Z), (B, Y)*and* (C, X). *We look now for a NE in mixed strategies of the form*

$$\begin{aligned} \sigma_1 &= aA + bB + (1 - a - b)C \\ \sigma_2 &= xX + yY + (1 - x - y)Z \end{aligned}$$

The expected utilities of player 1 when he chooses pure actions are

$$\begin{aligned} u_1(A, \sigma_2) &= 10x + 20(1 - x - y) = 20 - 10x - 20y \\ u_1(B, \sigma_2) &= 15x + 10y + 10(1 - x - y) = 10 + 5x \\ u_1(C, \sigma_2) &= 25x + 5y + 5(1 - x - y) = 5 + 20x \end{aligned}$$

The expected utilities of player 2 when he chooses pure actions are

$$\begin{aligned} u_2(\sigma_1, X) &= 30a + 35b + 25(1 - a - b) = 25 + 5a + 10b \\ u_2(\sigma_1, Y) &= 20a + 40b + 25(1 - a - b) = 25 - 5a + 15b \\ u_2(\sigma_1, Z) &= 30a + 40b + 25(1 - a - b) = 25 + 5a + 15b \end{aligned}$$

Thus, the expected payoffs of the players under these strategies are,

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= 5 + 20x + 15a - 20ya - 30xa + 5b - 15xb \\ u_2(\sigma_1, \sigma_2) &= 25 + 5a - 10ya + 15b - 5xb \end{aligned}$$

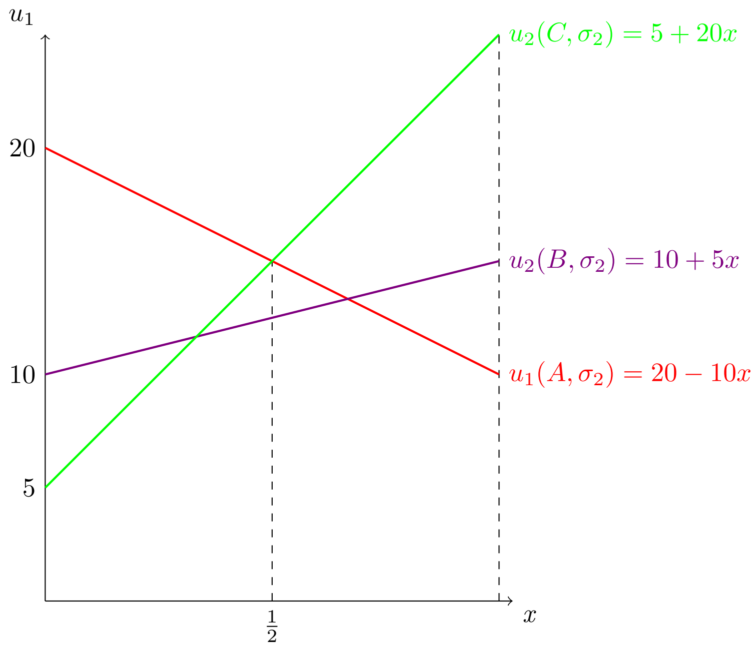
Case 1: Find all NE with $\mathbf{a > 0}$. *We start by noting that if* $a > 0$, *then strategy* Y *is dominated by strategy* Z *for player 2. Thus, we may assume that*

$$y = 0$$

and we obtain

$$\begin{aligned} u_1(A, \sigma_2) &= 20 - 10x \\ u_1(B, \sigma_2) &= 10 + 5x \\ u_1(C, \sigma_2) &= 5 + 20x \end{aligned}$$

We represent this three functions graphically,



Hence,

$$\text{BR}_1(\sigma_2) = \begin{cases} A \quad (a=1, b=0) & \text{if } x < \frac{1}{2} \\ \{A, C\} \quad (a \in [0,1], b=0) & \text{if } x = \frac{1}{2} \\ C(a=0, b=0) & \text{if } x > \frac{1}{2} \end{cases}$$

Since we are assuming $a > 0$*, we do not study, for the moment, the case* $x > \frac{1}{2}$*. We may assume*

$$b = 0$$

Hence,

$$\begin{aligned} u_2(\sigma_1, X) &= 25 + 5a \\ u_2(\sigma_1, Z) &= 25 + 5a \end{aligned}$$

and player 2 is indifferent between X *and* Z *for any value of* $0 \le a \le 1$*. Hence, for* $a > 0$*, we obtain the following two families of NE.*

1.

$$\begin{aligned} s_1 &= A \\ \sigma_2 &= xX + (1-x)Z, \quad 0 \le x < \frac{1}{2} \end{aligned}$$

with payoffs $u_1 = 20 - 10x$*,* $u_2 = 30$*.*

2.

$$\begin{aligned} \sigma_1 &= aA + (1-a)C, \quad 0 < a \le 1 \\ \sigma_2 &= \frac{1}{2}X + \frac{1}{2}Z \end{aligned}$$

The (expected) utilities of the players are

$$u_1(\sigma_1, \sigma_2) = 15, \quad u_2(\sigma_1, \sigma_2) = 25 + 5a$$

Case 2: Find all NE with $\mathbf{a=0}$. *If $a=0$ we have that the expected utilities of player 2 when he chooses pure actions are*

$$\begin{aligned} u_2(\sigma_1, X) &= 25+10b \\ u_2(\sigma_1, Y) &= 25+15b \\ u_2(\sigma_1, Z) &= 25+15b \end{aligned}$$

So for $b>0$, strategy X is dominated by strategies Y and Z, for player 2. Le us assume $b>0$. Then we may also assume $x=0$. That is, $\sigma_2 = yY+(1-y)Z$. And we obtain

$$\begin{aligned} u_1(A, \sigma_2) &= 20-20y \\ u_1(B, \sigma_2) &= 10 \\ u_1(C, \sigma_2) &= 5 \end{aligned}$$

Since, we assume $a=0$, we see that

$$B = \mathrm{BR}_1(\sigma_2)$$

iff $y \geq \frac{1}{2}$. and we obtain the family of NE

$$\begin{aligned} s_1 &= B \\ \sigma_2 &= yY+(1-y)Z, \quad \frac{1}{2} \leq y \leq 1 \end{aligned}$$

with payoffs $u_1 = 10$, $u_2 = 40$.

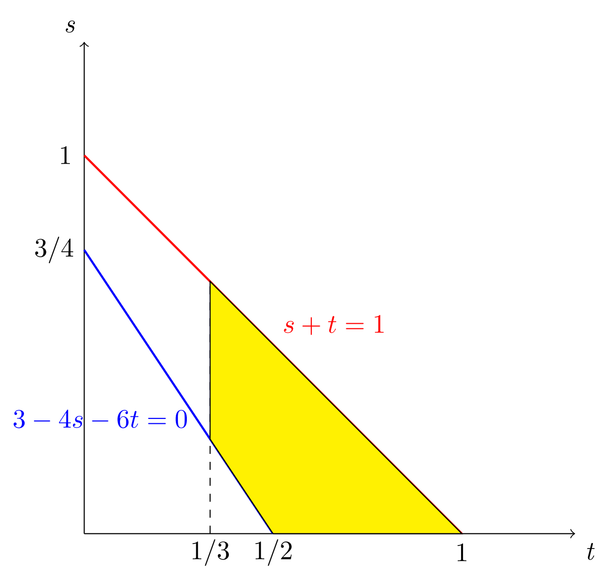
For $a=b=0$, player 2 is indifferent between all strategies X, Y, Z. Player 1 chooses $a=b=0$ iff

$$\begin{aligned} u_1(A, \sigma_2) &= 20-10x-20y \leq 5+20x = u_1(C, \sigma_2) \\ u_1(B, \sigma_2) &= 0+5x \leq 5+20x = u_1(C, \sigma_2) \end{aligned}$$

That is, if

$$x \geq \frac{1}{3}, \quad 3 \leq 6x+4y$$

Graphically,



The yellow area represents the values of (x,y) for which we have $3-4y-6x \leq 0$ and $1-3x \leq 0$. This region may be described as

$$A = \{(x,y) \in \mathbb{R}^2 : \frac{1}{3} \leq x \leq \frac{1}{2}, \quad \frac{3}{4} - \frac{3}{2}x \leq y \leq 1-x\} \cup \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} \leq x \leq 1, \quad 0 \leq y \leq 1-x\}$$

Thus, we find infinitely many NE of the form

$$\sigma_1 = C$$
$$\sigma_2 = xX + yY + (1-x-y)Z, \quad x, y \in A$$

The utilities of the players are

$$u_1 = 5 + 20x \quad u_2 = 25$$

Exercise 4: Consider a game in which there is a prize worth \$30. There are three contestants, A, B, and C. Each can buy a ticket worth \$15 or \$30, or not buy a ticket at all. They make these choices simultaneously and independently. Once the ticket purchase decisions have been made, the prize is awarded. If no one bought the ticket, the prize is not awarded. Otherwise, the ticket is awarded to the buyer of the higher cost ticket if there is only one such player, or split equally between two or three players if there are ties among the highest cost ticket buyers.

1. Write the normal form of the game. (Hint: Draw a box for each of player C?s possible strategies.)
2. Identify the Nash equilibria in pure strategies of this game.

Solution: *Player C chooses one of the following 3 matrices*

(a) Player C chooses 0:

Player A \ Player B	0	15	30
0	$0, 0, 0$	$0, 15, 0$	$0, 0, 0$
15	$15, 0, 0$	$0, 0, 0$	$-15, 0, 0$
30	$0, 0, 0$	$0, -15, 0$	$-15, -15, 0$

(b) Player C chooses 15:

Player A \ Player B	0	15	30
0	$0, 0, 15$	$0, 0, 0$	$0, 0, -15$
15	$0, 0, 0$	$-5, -5, -5$	$-15, 0, -15$
30	$0, 0, -15$	$0, -15, -15$	$-15, -15, -15$

(c) Player C chooses 30:

Player A \ Player B	0	15	30
0	$0, 0, 0$	$0, -15, 0$	$0, -15, -15$
15	$-15, 0, 0$	$-15, -15, 0$	$-15, -15, -15$
30	$-15, 0, -15$	$-15, -15, -15$	$-20, -20, -20$

The best responses of the players are

(a) Player C chooses 0:

Player A \ Player B	0	15	30
0	$0, 0, 0$	$0, 15, 0$	$0, 0, 0$
15	$15, 0, 0$	$0, 0, 0$	$-15, 0, 0$
30	$0, 0, 0$	$0, -15, 0$	$-15, -15, 0$

(b) Player C chooses 15:

Player A \ Player B	0	15	30
0	$0, 0, 15$	$0, 0, 0$	$0, 0, -15$
15	$0, 0, 0$	$-5, -5, -5$	$-15, 0, -15$
30	$0, 0, -15$	$0, -15, -15$	$-15, -15, -15$

(c) Player C chooses 30:

Player A \ Player B	0	15	30
0	$0, 0, 0$	$0, -15, 0$	$0, -15, -15$
15	$-15, 0, 0$	$-15, -15, 0$	$-15, -15, -15$
30	$-15, 0, -15$	$-15, -15, -15$	$-20, -20, -20$

We see that there are 6 NE in pure strategies of to types:

Type 1: One player plays 0 *and the other two play* 15.

Type 2: One player plays 15 *and the other two play* 0.

Exercise 5: Two firms, 1 and 2, produce heterogeneous products. If the two firms set prices p_1 and p_2, respectively, the quantities demanded of the products of the two firms will be,

$$\begin{aligned} x_1(p_1,p_2) &= \max\left\{0, 180 - p_1 - \left(p_1 - \frac{p_1+p_2}{2}\right)\right\} \\ x_2(p_1,p_2) &= \max\left\{0, 180 - p_2 - \left(p_2 - \frac{p_1+p_2}{2}\right)\right\} \end{aligned}$$

The above demand functions describe a situation in which the products are not perfectly homogeneous. For instance, the demand for good 1, even if decreasing in the price set by firm 2, is positive also when $p_1 > p_2$ (which is not the case when the good are homogeneous), or, in a different way, the demand for good 1 will be lower than $180 - p_1$ when the price of good 1 is higher than the mean of the prices (and therefore higher than p_2) and higher than that when the price of good 1 is lower than the mean of the prices (and therefore lower than p_2). Assume that the firms have constant marginal costs $\text{MC}_1 = 20$ and $\text{MC}_2 = 20$. Assume also that the characteristics of the market are such that the firms have to set simultaneously the prices.

(a) Write down firms' profits, $\pi_1(p_1,p_2)$ and $\pi_2(p_1,p_2)$, as functions of the prices they set.

(b) Find firm 1's best response to a price set by firm 2, $\text{BR}_1(p_2)$.

(c) Find firm 2's best response to a price set by firm 1, $\text{BR}_2(p_1)$.

(d) Find the Nash equilibrium (p_1^*, p_2^*).

(e) Determine equilibrium profits, $\pi_1(p_1,^* p_2^*)$ and $\pi_2(p_1^*, p_2^*)$.

(f) Are equilibrium profits zero? Why?

(g) Draw the best response function and the Nash equilibrium in the $p_2 - p_1$ plane (p_2 on the horizontal axis and p_1 on the vertical axis).

Solution:

(a) We assume prices are such that the quantities demanded are positive. The profits of the firms are

$$\begin{aligned} \pi_1(p_1,p_2) &= \left(180 - p_1 - \left(p_1 - \frac{p_1+p_2}{2}\right)\right)(p_1 - 20) = -\frac{3p_1^2}{2} + \frac{p_1p_2}{2} + 210p_1 - 10p_2 - 3600 \\ \pi_2(p_1,p_2) &= \left(180 - p_2 - \left(p_2 - \frac{p_1+p_2}{2}\right)\right)(p_2 - 20) = \frac{p_1p_2}{2} - 10p_1 - \frac{3p_2^2}{2} + 210p_2 - 3600 \end{aligned}$$

(b) Firm 1's best response to a price set by firm 2 is

$$\text{BR}_1(p_2) = \frac{p_2 + 420}{6}$$

(c) Fiirm 2's best response to a price set by firm 1 is

$$\text{BR}_2(p_1) = \frac{p_1 + 420}{6}$$

(d) *The Nash equilibrium* (p_1^*, p_2^*) *is the solution of the following system of equations*

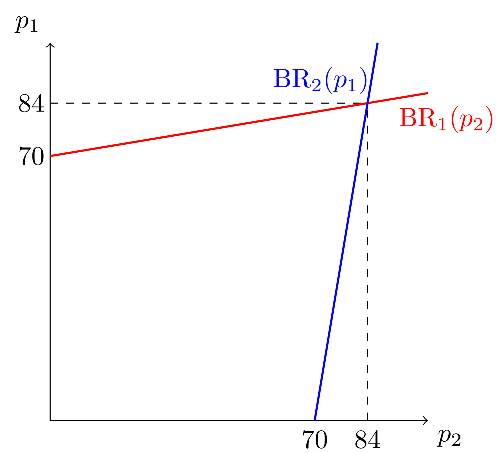
$$\begin{aligned} p_1 &= \frac{p_2 + 420}{6} \\ p_2 &= \frac{p_1 + 420}{6} \end{aligned}$$

The solution is $p_1^* = p_2^* = 84$.

(e) *The equilibrium profits are* $\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = 6144$.

(f) *The equilibrium profits aren't zero because products are heterogeneous.*

(g) *Graphically, the best response function and the Nash equilibrium are*



Exercise 6: Two firms, 1 and 2, produce heterogeneous products. If the two firms set prices p_1 and p_2, respectively, the quantities demanded of the products of the two firms will be:

$$\begin{aligned} x_1(p_1, p_2) &= 140 - p_1 - \left(p_1 - \frac{p_1 + p_2}{2}\right) \\ x_2(p_1, p_2) &= 140 - p_2 - \left(p_2 - \frac{p_1 + p_2}{2}\right) \end{aligned}$$

Suppose the two firms have constant marginal costs MC1 = 40 and MC2 = 40 and that they have to set their prices simultaneously.

(a) What is the Nash equilibrium ? What are the Nash equilibrium profits for firms 1 and 2?

(b) Are the profits in equilibrium equal to 0? Why or why not?

Solution:

(a) *We assume prices are such that the quantities demanded are positive. The profits of the firms are*

$$\begin{aligned} \pi_1(p_1, p_2) &= \left(140 - p_1 - \left(p_1 - \frac{p_1 + p_2}{2}\right)\right)(p_1 - 40) = -\frac{3p_1^2}{2} + \frac{p_1 p_2}{2} + 200p_1 - 20p_2 - 5600 \\ \pi_2(p_1, p_2) &= \left(140 - p_2 - \left(p_2 - \frac{p_1 + p_2}{2}\right)\right)(p_2 - 40) = \frac{p_1 p_2}{2} - 20p_1 - \frac{3p_2^2}{2} + 200p_2 - 5600 \end{aligned}$$

Firm 1's best response to a price set by firm 2 is

$$\mathrm{BR}_1(p_2) = \frac{p_2 + 400}{6}$$

Fiirm 2's best response to a price set by firm 1 is

$$\text{BR}_2(p_1) = \frac{p_1 + 400}{6}$$

The Nash equilibrium (p_1^*, p_2^*) is the solution of the following system of equations

$$\begin{aligned} p_1 &= \frac{p_2 + 400}{6}. \\ p_2 &= \frac{p_1 + 400}{6} \end{aligned}$$

The solution is $p_1^* = p_2^* = 80$. The equilibrium profits are $\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = 2400$.

(b) The equilibrium profits aren't zero because products are heterogeneous.

Exercise 7: Assume that two firms, 1 and 2, produce heterogeneous products and the quantities demanded by the market, when these firms fix prices p_1 and p_2, are, respectively:

$$\begin{aligned} x_1(p_1, p_2) &= 200 - p_1 + p_2/3 \\ x_2(p_1, p_2) &= 200 - p_2 + p_1/3 \end{aligned}$$

These demand functions describe a situation in which products are not perfectly homogenous. Suppose that both firms have constant marginal costs MC= 50 and MC2 = 50 and that the features of the market are such that both firms have to set prices simultaneously.

(a) Find the Nash equilibrium (p_1^*, p_2^*).

(b) Determine equilibrium profits, $\pi_1(p_1^*, p_2^*)$ and $\pi_2(p_1^*, p_2^*)$. Are equilibrium profits zero? Why?

(c) Suppose now that firm 1 and 2 could make an agreement on how to make their products different so that the demand functions become,

$$\begin{aligned} x_1(p_1, p_2) &= 200 - p_1 + p_2/2 \\ x_2(p_1, p_2) &= 200 - p_2 + p_1/2 \end{aligned}$$

Do you think that it pays for the firms to agree on these new differentiated products?

Solution: *Let us consider the following market demand functions*

$$\begin{aligned} x_1(p_1, p_2) &= 200 - p_1 + bp_2 \\ x_2(p_1, p_2) &= 200 - p_2 + bp_1 \end{aligned}$$

with $0 \le b < 2$. We assume prices are such that the quantities demanded are positive. Now, the profits of the firms are

$$\begin{aligned} \pi_1(p_1, p_2) &= (200 - p_1 + bp_2)(p_1 - 50) = bp_1p_2 - 50bp_2 - p_1^2 + 250p_1 - 10000 \\ \pi_2(p_1, p_2) &= (200 - p_2 + bp_1)(p_2 - 50) = bp_1p_2 - 50bp_1 - p_2^2 + 250p_2 - 10000 \end{aligned}$$

Firm 1's best response to a price set by firm 2 is

$$\text{BR}_1(p_2) = \frac{1}{2}(bp_2 + 250)$$

Fiirm 2's best response to a price set by firm 1 is

$$\text{BR}_2(p_1) = \frac{1}{2}(bp_1 + 250)$$

The Nash equilibrium (p_1^*, p_2^*) *is the solution of the following system of equations*

$$\begin{aligned} p_1 &= \frac{1}{2}(bp_2 + 250) \\ p_2 &= \frac{1}{2}(bp_1 + 250) \end{aligned}$$

The solution is

$$p_1^* = p_2^* = \frac{250}{2-b}$$

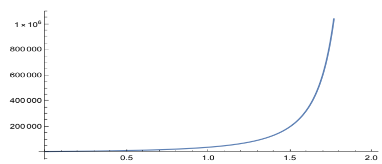
The equilibrium profits are

$$\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = \frac{2500(b+3)^2}{(2-b)^2}$$

Note that the funcion

$$f(b) = \frac{2500(b+3)^2}{(2-b)^2}$$

is increasing in b. *Its graph is the following.*



(a) *Taking*

$$b = \frac{1}{3}$$

we obtain that the Nash equilibrium (p_1^*, p_2^*) *is is* $p_1^* = p_2^* = 150$.

(b) *Taking*

$$b = \frac{1}{3}$$

the equilibrium profits are $\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = 10000$. *The equilibrium profits aren't zero because products are heterogeneous.*

(c) *Suppose now that firm 1 and 2 could make an agreement on how to make their products different so that the demand functions become,*

$$\begin{aligned} x_(p_1, p_2) &= 200 - p_1 + p_2/2 \\ x_2(p_1, p_2) &= 200 - p_1 + p_1/2 \end{aligned}$$

Now, taking

$$b = \frac{1}{2}$$

The Nash equilibrium is $p_1^* = p_2^* = \frac{460}{3}$. *And the equilibrium profits are* $\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = \frac{96100}{9} \approx 10677.61$. *Hence, the firms would benefit from product differentiation.*

Exercise 8: Firms 1 and 2 compete in a market with a homogeneous product. Firms have to decide how much to produce and put up for sale. Let q_1 and q_2 denote the quantities produced and put up for sale by firm 1 and 2, respectively. The clearing market price of the homogeneous product on sale depends on quantities produced according to the following inverse demand function: $P(Q) = \max\{0, 10 - Q\}$, where $Q = q_1 + q_2$ is the total number of units of the product in the market. Total production costs are $c_1(q_1) = q_1^2$ and $c_2(q_2) = q_2^2$. This is the Cournot model of oligopoly competition that you will study in the course of Industrial Organization.

(a) Describe the situation as a normal form game.

(b) Find the Nash equilibrium.

Solution:

(a) The set of players is $N = \{1, 2\}$, the sets of strategies are $S_1 = S_2 = [0, \infty)$ and the payoffs are

$$\begin{aligned}\pi_1(q_1, q_2) &= (10 - q_1 - q_2)\, q_1 - q_1^2 \\ \pi_2(q_1, q_2) &= (10 - q_1 - q_2)\, q_2 - q_2^2\end{aligned}$$

(b) The profits of the firms are

$$\begin{aligned}\pi_1(q_1, q_2) &= (10 - q_1 - q_2)\, q_1 - q_1^2 = q_1(-q_1 - q_2 + 10) - q_1^2 \\ \pi_2(q_1, q_2) &= (10 - q_1 - q_2)\, q_2 - q_2^2 = q_2(-q_1 - q_2 + 10) - q_2^2\end{aligned}$$

Firm 1's best response to a price set by firm 2 is

$$\mathrm{BR}_1(q_2) = \frac{10 - q_2}{4}$$

Firm 2's best response to a price set by firm 1 is

$$\mathrm{BR}_2(q_1) = \frac{10 - q_1}{4}$$

The Nash equilibrium (q_1^*, q_2^*) is the solution of the following system of equations

$$\begin{aligned}q_1 &= \frac{10 - q_2}{4} \\ q_2 &= \frac{10 - q_1}{4}\end{aligned}$$

The solution is $q_1^* = q_2^* = 2$. The Equilibrium profits are $\pi_1^* = \pi_2^* = 8$.

Exercise 9: Two Airlines, Fly and Connect, sell a round trip in economic class between two cities for \$400. Each one of the Airlines serves half the market and it thinks that if it lowers its fare by a 10% and the competitor does not change its fare, its demand would rise by a 40% (which means that its price elasticity is -4). Of this increase, 20% represents clients of the other company who change attracted by lower fares, and the remaining 20% represent new clients attracted by lower fares. If both companies lower their price by 10% both are going to increase by a 20 their respective demands due to new clients (price elasticity is -2). The unitary cost for carrying each passenger is \$200 for each of the Airlines. Assume that they have to simultaneously decide whether to lower the prices by 10% or not.

(a) Represent the normal form of the game and find the Nash equilibrium.

Solution: *Let $2N$ be the total number of customers. Note that the customers each company gets*

		Connect	
		\$ 400	*\$ 360*
Fly	*\$ 400*	N, N	$N(1-0.2), N(1+0.4)$
	\$ 360	$N(1+0.4), N(1-0.2)$	$N(1+0.2), N(1+0.2)$

Number of passengers

So the total profits are

		Connect: \$ 400	Connect: \$ 360
Fly	\$ 400	$N(400-200), N(400-200)$	$N(1-0.2)(400-200), N(1+0.4)(360-200)$
	\$ 360	$N(1+0.4)(360-200), N(1-0.2)(400-200)$	$N(1+0.2)(360-200), N(1+0.2)(360-200)$

Number of passengers

That is,

		Connect: \$ 400	Connect: \$ 360
Fly	\$ 400	$200N, 200N$	$160N, 224N$
	\$ 360	$224N, 160N$	$192N, 192N$

Normalized profits

By normalizing $N = 1$, we see that the normal form of the game is the following

		Connect: \$ 400	Connect: \$ 360
Fly	\$ 400	$200, 200$	$160, 224$
	\$ 360	$224, 160$	$192, 192$

Normalized profits

The NE is $(\$360, \$360)$ *with payoffs* $u_1 = u_2 = 192$.

(b) Assume now that the two Airlines agree to have their total revenues accrue into a common pool which is then shared equally, while each airline pays its own costs. As before, the Airlines have to decide whether or not lowering their fares by 10%. Represent the game in its normal form and find the Nash equilibrium.

Solution: *Let $2N$ be the total number of customers. Note that the customers each company gets*

		Connect: \$ 400	Connect: \$ 360
Fly	\$ 400	N, N	$0.8N, 1.4N$
	\$ 360	$1.4N, 0.8N$	$1.2N, 1.2N$

Number of passengers

So, the total revenue for each company is

		Connect: \$ 400	Connect: \$ 360
Fly	\$ 400	$400N, 400N$	$0.8 \times 400N, 1.4 \times 360N$
	\$ 360	$1.4 \times 360N, 0.8 \times 400N$	$1.2 \times 360N, 1.2 \times 360N$

total revenue

That is,

		Connect: \$ 400	Connect: \$ 360
Fly	\$ 400	$400N, 400N$	$320N, 504N$
	\$ 360	$504N, 320N$	$432N, 432N$

total revenue

Total revenue for both companies are

		Connect	
		\$ 400	*\$ 360*
Fly	*\$ 400*	$800N$	$824N$
	\$ 360	$824N$	$864N$

total revenue

And the profits are (since they share the revenue),

		Connect	
		\$ 400	*\$ 360*
Fly	*\$ 400*	$400N - 200N, 400N - 200N$	$412N - 200 \times 0.8 \times N, 412N - 200 \times 1.4 \times N$
	\$ 360	$412N - 200 \times 1.4 \times N, 412N - 200 \times 0.8 \times N$	$432 - 200 \times 1.2 \times N, 432 - 200 \times 1.2 \times N$

Profits

That is,

		Connect	
		\$ 400	*\$ 360*
Fly	*\$ 400*	$200N, 200N$	$252N, 132N$
	\$ 360	$132N, 252N$	$192N, 192N$

profits

By normalizing $N = 1$, *we see that the normal form of the game is the following*

		Connect	
		\$ 400	*\$ 360*
Fly	*\$ 400*	$200, 200$	$252, 132$
	\$ 360	$132, 252$	$192, 192$

Normalized profits

The NE (in dominant strategies) is $(\$400, \$400)$ *with payoffs* $u_1 = u_2 = 200$.

(c) Do you believe that companies are going to act as in (b). Why?

Solution: *Yes. The common pool reduce the incentives to lower prices and lead to high equilibrium prices.*

Exercise 10: Two theaters are located on the same street. They advertise and, by so doing, attract customers to the area so that each of the two theaters' advertising is beneficial to both. Let x_1 be the advertising level of theater 1 and x_2 be the advertising level of theater 2 (for instance, x_1 could denote the number of potential customers reached by theater 1's advertising). The profit functions for theaters 1 and 2 are respectively:

$$\begin{aligned} \pi_1(x_1, x_2) &= (30 + x_2)x_1 - 2x_1^2 \\ \pi_2(x_1, x_2) &= (30 + x_1)x_2 - 2x_2^2 \end{aligned}$$

(a) What are the Nash equilibrium advertising levels of the two theaters?

(b) What are their equilibrium profits?

(c) Could the theaters increase total profits if they could commit to given advertising levels? What advertising levels would they choose?

Exercise 11: For most new films, the demand for movie theaters is highest in the first few days after opening, then taper off. Accordingly, studios must time new releases very carefully. Two key factors affecting potential demand are the season and the timing of other releases. Suppose that both Suprafilm and National are producing major action movies. The two studios simultaneously must choose between release in November or December. If both films open on November, each will sell 200,000 tickets. If one opens on November and the other in December, then the early release will sell 350,000 tickets, while the later release will sell 220,000. If both open in December, each will sell 300,000 tickets.

(a) Construct a game in normal form to illustrate the situation.

(b) Identify the Nash equilibrium or equilibria.

Exercise 12: Two executives, A and B, have to submit a proposal to a firm's board about the price of the firm's product. There are four possible prices: 100, 110, 120 or 130 euros. The board is made up of 13 members and conversations prior to the meeting have led four members to prefer the price of 100, two members to prefer the price of 110, one member prefers 120, and six members prefer the price of 130. Both executives have to submit their proposals simultaneously. Once the proposals are received, each member of the board will vote for one of the executives as the next CEO. Each member will vote for the executive whose price proposal is closer to his own ideal price. If both executives propose the same price, members vote with probability 1/2 for each. If one member has to decide between two equidistant prices (from his ideal price), he will vote for each executive with probability 1/2. The executive with more votes will be the next CEO.

(a) Design a table with the (expected) votes each executive would get depending on both proposals for prices. (Suggestion: Place executive A's proposed prices in rows and B's prices in columns) Suppose that both executives only care about being chosen as CEO and that they do not have any other preference regarding a particular price.

(b) Find the normal form of this game.

(c) Find the pure strategy Nash equilibria.

(d) Comment on the result.

Exercise 13: Two firms, H and S, produce complementary hardware and software products. This means that if one firm dedicates more resources to investment, it increases not only the demand of its own product but also the demand of the product of the other firm. Letting x_H and x_S denote the investment expenditures of H and S, respectively, their profits are

$$\begin{aligned}\pi_H(x_H, x_S) &= (30 - x_S)x_H + 40x_S - \frac{x_H^2}{2} \\ \pi_S(x_H, x_S) &= (30 - x_H)x_S + 40x_H - \frac{x_S^2}{2}\end{aligned}$$

(a) If firms H and S simultaneously decide their investment expenditures, what are the equilibrium investment expenditures?

(b) What are the equilibrium profits?

(c) Suppose now that firm H can make a take-it-or-leave-it offer to buy firm S from its owners. What is the lowest amount that the owners of firm S would be willing to accept?

(d) Suppose that the owners of firm S accept the offer, and that the new firm will decide both x_H and x_S. What are the optimal investment expenditures for the new firm? What profit will it obtain?

(e) Suppose that firm H makes the lowest offer that would be accepted by the owners of firm S, that they accept such offer and that the new firm will then set the optimal investment expenditure levels that you have found in part (d). Would the profits of firm H (net of the price paid to buy firm S) increase? If so, by how much?

Exercise 14: The three members of the board of directors of a corporation, A, B, and C, have to vote on whether to raise their salaries. Assume that each director prefers a raise over no raise; however, each also knows that voting for a pay raise may reduce the probability of being reappointed at the end of his/her term. Thus, the best possible outcome for an individual director is to vote against a pay raise that still wins majority approval, and the worst possible outcome is to vote for a pay rise that fails. Not all directors agree on how to rank the other two outcomes. For directors B and C, the second-best outcome is to have a raise proposal pass with their vote being in favor. For director A, however, the second-best outcome is for a raise proposal to fail, with his vote being against. The payoff ranking below (where large numbers represent more favorable outcomes) summarizes this situation. (You should assume that each director knows the rankings of all three directors.) Assume that the directors vote simultaneously.

Outcomes	**Director A's rankings**	**Director B's and C's rankings**
Best, 3 points	Raise passes, vote against	Raise passes, vote against
Second best,	Raise fails, vote against points	Raise passes, vote for
Third best, 1 point	Raise passes, vote for	Raise fails, vote against
Worst, 0 points 1 point	Raise fails, vote for	Raise fails, vote for

(a) Write the normal form of this game (Suggestion: Let A be the row player, B be the column player and C be the matrix player.)

(b) Find the Nash equilibria of this game.

(c) Explain (in words) the results of part (b).

Exercise 15: The board of directors of a company is composed of 9 directors. The board of directors has to make a decision about the CEO and there are three possible choices

A: Fire the CEO .

B: Keep the CEO but not give him a pay raise.

C: Keep the CEO and give him a pay raise.

Directors have different opinions about the appropriate course of actions. Some would want to fire the CEO and get a new CEO on board. Some believe that the CEO should not be fired, but that there is no need to give him a pay raise. Some finally believe that the CEO should be kept, but that only a pay raise will make sure that he will not be snatched away by the competition.

The ordering of preferences is the following:

- Four directors (the 'archenemies') prefer A to B to C
- Three directors (the 'enthusiasts') prefer C to B to A
- Two directors (the 'moderate') prefer B to C to A

The board of directors has agreed to the following voting procedure. Each director may vote for one of the three choices. Then, the choice with the least number of votes is dropped, and the directors vote on the remaining two. Whichever choice receives the most votes in round two is the winner.

(a) What would happen if each director voted according to his true preferences?

(b) Consider now the case in which the directors are aware of the strategic implications of their actions. How would you vote if you were an archenemy? [Hint 1: Do not try to draw the extensive form because it is very complicated and it will not help you much. Hint 2: Ask yourself: 1) What are the possible outcomes of the first vote? 2) For each of these possible outcomes, how would directors vote in the second vote? 3) Given the answers to the previous questions, how do you think each type of director will vote in the first vote?]

Exercise 16: Two bidders, A and B, participate in an auction to buy a plot of land. Each of the two bidders can only bid one of three possible prices, low, medium and high. The plot will be sold to the highest bidder and in case of a tie a coin will be tossed to determine the winner.

The value of obtaining the plot net of the price to be paid is the following:

Price	**Net value**
High	10
Medium	30
Low nt	40

(a) Write down the normal form of the game.

(b) Find the pure strategy Nash equilibria.

Consider now the possibility of modifying the rules of the auction so that in case of a tie the winner is A with probability 1.

(c) Write down the normal form of the game.

(c) Find the pure strategy Nash equilibria.

(c) If you wanted to sell the plot and therefore wanted to maximize the expected sale price, which of the two auction formats would you prefer? Explain your answer.