Chapter 2
Static Games with asymmetric information

October 18, 2024

1 The Bayesian equilibrium.

Definition 1.1. A Bayesian game $G = (\Omega, \mu, A, N, T, \{p_i\}_{i=1}^n, \{u_i\}_{i=1}^n)$ consists of the following,

- A set of *states* Ω and a common prior μ on Ω.
- The set of *players* $N = \{1, 2, \ldots, n\}$.
- For each player $i \in N$,
  - The of *actions* A_i of the player. $A = A_1 \times A_2 \times \cdots \times A_n$, $a = (a_1, a_2, \ldots, a_n)$
  - The set of possible *types* T_i for the player. The type $t_i \in T_i$ is known only by player $i = 1, 2, \ldots, n$. The set of all the types is $T = T_1 \times T_2 \times \cdots \times T_n$. We use the notation $t = (t_1, t_2, \ldots, t_n)$, $t_{-i} = (t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots t_n)$.
  - A mapping $\tau_i : \Omega \to T_i$, that represents what is know by player i.
  - The *beliefs* $p_i : 2^\Omega \times T_i) \to [0, 1]$ of player i such that

$$p_i(A|t_i) = \frac{\mu\left(A \cap \tau_i^{-1}(t_i)\right)}{\mu\left(\tau_i^{-1}(t_i)\right)}$$

    where $\tau_i^{-1}(t_i) = \{\omega \in \Omega : \tau_i(\omega) = t_i\}$.

Typically, $\Omega = T_1 \times \cdots \times T_n$ and τ_i is the projection mapping. It is interpreted as the information that agent i has on the types of the other agents (t_{-i}), given that his type is t_i.

- A **pure strategy** of player i,

$$s_i : T_i \to A_i$$

  specifies the action of each of player for each of his possible types i.

- A **mixed strategy** of player i,

$$\sigma_i : T_i \to \Delta(A_i)$$

  is a vector $\sigma_i(t_i) = (\sigma_i(a_1, t_i), \ldots, \sigma_i(a_n, t_i))$. Here, $\sigma_i(a_k, t_i)$ is the probability that agent i of type t_i plays the strategy a_k.

- *The funcions of utility over outcomes:* of the agents $u_1, u_2, \ldots, u_n$ are of the form

$$u_i : A \times T \to \mathbb{R}$$

  $u_i(a; t)$ depends on the types $t = (t_1, \ldots, t_n)$ and on the actions of all the agents $a = (a_1, \ldots, a_n)$.

- *Utility functions over strategies:* The expected utility of agent i when he plays strategy s_i, given that the other agents are playing the pure strategies s_{-i} is

$$\sum_{w \in \Omega} u_i\big(s_1(\tau_1(\omega)), \ldots, s_n(\tau_n(\omega))\big)\mu(w)$$

- The notion of equilibrium is the NE with the above expected utility functions.

2 Examples.

2.1 Bank runs

Bank runs

Two players deposit €100 each in a bank. If the bank manager is a good investor both players get €150 at the end of the year. If not both players loose their money. Both players may try to withdraw the money now. But the bank has only €100 cash. If only one tries to withdraw she gets €100. If both try to withdraw each gets €50. Player 1 believes that the manager is good with probability q. Player 2 knows whether the manager is good or bad. Player 1 and player 2 simultaneously decide whether to withdraw or not.

The situation can be described with the following two tables.

			P2	
			y	$1-y$
			W	N
P1	x	W	50 , 50	100 , 0
	$1-x$	N	$0, 100$	150 , 150

good q

			P2	
			z	$1-z$
			W	N
P1	x	W	50 , 50	100 , 0
	$1-x$	N	$0, 100$	0 , 0

bad $1-q$

We first describe this as Bayesian Game.

- $\Omega = \{g, b\}$, where g stands for the manager is good and b stands for the manager is bad.
- $\mu(g) = q$, $\mu(b) = 1 - q$.
- $N = \{1, 2\}$.
- $A_1 = A_2 = \{W, N\}$.
- $T_1 = \{1\}$, $T_2 = \{g, b\}$.
- $\tau_1(g) = \tau_1(b) = 1$, $\tau_2(g) = g$, $\tau_2(b) = b$.
- Beliefs of player 1. $\mu\left(\tau_1^{-1}(1)\right) = \mu(\Omega) = 1$, $\mu\left(a \cap \tau_1^{-1}(1)\right) = \mu(a \cap \Omega) = \mu(a) = q$

$$p_1(g|t_1) = q p_1(b|t_1) = 1 - q$$

The way we solve this game is that there are three players:

- 'two types' of player 2, $T_2 = \{g, b\}$, where g stands for the type of player 2 who knows that the manager is good. and b stands for the type of player 2 who knows that the manager is bad.
- and one type of player 1 $T_1 = \{1\}$, the one who is not informed of the situation.

Player 1 does not know if she is facing player b or player g. She 'thinks' that she is facing player g with probability q and player b with probability $1 - q$. That is her beliefs are

$$p_1(g|1) = q, \quad p_1(b|1) = 1 - q, \quad p_2(1|g) = p_2(1|b) = 1$$

In this course we will simply write

$$p_1(g) = q, \quad p_1(b) = 1 - q$$

We look for a BNE in mixed por pure strategies of the form

- $\sigma_1 = xW + (1-x)N$
- $\sigma_g = yW + (1-y)N$
- $\sigma_b = zW + (1-z)N$

Observation 2.1. In many books you will see the notation $\sigma_g = \sigma_g(*|g)$, $\sigma_b = \sigma_g(*|b)$ and so on. This is actually the proper notation. But, for the purposes of this course, in which we analyze 'easy games' we will follow the above shorter notation.

Let us start with the best replies of player 2. We have

$$\begin{array}{cc} \mathrm{BR}_g(W) = W & \mathrm{BR}_g(N) = W \\ \mathrm{BR}_b(W) = N & \mathrm{BR}_b(N) = W \end{array}$$

which we shorten to

$$\mathrm{BR}_2(W) = WW \quad \mathrm{BR}_2(N) = NW$$

Note that player b has a dominant strategy, W. In particular, player b will not use a mixed strategy in a BNE. That is any BNE must be of the form

- $\sigma_1 = xW + (1-x)N$
- $\sigma_g = yW + (1-y)N$
- $\sigma_b = W$

Also, any BNE in pure strategies must be of the

$$(*, WW) \quad \text{or} \quad (*, NW)$$

Let us look for a **pooling equilibrium** of the form $(*, WW)$. The expected utilities of player 1 are

$$\begin{aligned} u_1(W, WW) &= q50 + (1-q)50 = 50 \\ u_1(N, WW) &= q0 + (1-q)0 = 0 \end{aligned}$$

Thus, $\mathrm{BR}_1(WW) = W$ and we have that for every $0 \le q \le 1$, (W, WW) is a BNE, with payoffs $u_1 = 50$, $u_g = 50$, $u_b = 50$.

Let us look for a **separating equilibrium** of the form $(*, NW)$. The expected utilities of player 1 are

$$\begin{aligned} u_1(W, NW) &= q100 + (1-q)50 = 50 + 50q \\ u_1(N, NW) &= q150 + (1-q)0 = 150q \end{aligned}$$

We see that

- if $q < \frac{1}{2}$, then $\mathrm{BR}_1(NW) = W$. But, $\mathrm{BR}_2(W) = WW$, So there is no BNE of the form $(*, NW)$.
- if $q \ge \frac{1}{2}$, then $\mathrm{BR}_1(NW) = N$. Since, $\mathrm{BR}_2(N) = NW$ we have that (N, NW) is a BNE with payoffs $u_1 = 150q$, $u_g = 150$, $u_b = 100$.

Let us now look for a BNE in which $0 < x, y < 1$. Player g must be indifferent between W and N.

$$\begin{aligned} u_g(\sigma_1, W) &= 50x + 100(1-x) = 100 - 50x \\ u_g(\sigma_1, N) &= 0 \times x + (1-x)150 = 150 - 150x \end{aligned}$$

We must have $100 - 50x = 150 - 150x$, so $x = \frac{1}{2}$. The utilities of player 1 are

$$\begin{aligned} u_1(W, \sigma_2) &= q(50y + 100(1-y)) + (1-q)50 = 50 + 50q - 50qy \\ u_1(N, \sigma_2) &= q(0 \times y + 150(1-y)) + (1-q)0 = 150q - 150qy \end{aligned}$$

We must have $50 + 50q - 50qy = 150q - 150qy$, so

$$y = 1 - \frac{1}{2q}$$

And because $0 < y < 1$, this is valid if and only if $q > \frac{1}{2}$. Hence, we have that for $q > \frac{1}{2}$

$$\left(\frac{1}{2}W + \frac{1}{2}B, \left(\frac{1}{2}W + \frac{1}{2}B, N\right)\right)$$

is BNE with payoffs

$$u_1 = 50\left(q + \frac{1}{2q}\right), \quad u_2 = 75$$

2.2 Cournot competition with asymmetric information

Cournot Duopoly

- Suppose there are two companies which compete in quantities (Cournot's competion).
- The inverse demand is
$$P(q) = a - q$$
where
$$q = q_1 + q_2$$
and q_1, q_2 are the amounts produced by the companies.
- The cost function of Firm 1 is
$$c_1(q_1) = c_1 q_1$$
- The cost function of Firm 2 is
$$c_2(q_2) = \begin{cases} c_h q_2 & \text{with probability } \theta \\ c_l q_2 & \text{with probability } 1-\theta \end{cases}$$
with $c_l < c_h$.
- There is asymmetric information:
  - Firm 2 knows its cost cost function (c_h or c_l) and knows the cost of Firm 1.
  - But, Firm 1 only knows its cost function. It also knows that the marginal cost of Firm 2 is c_h with probability θ and c_l with probability $1-\theta$.
  - Firm 2 has more information than Firm 1.
- First we represent this situation as a **Bayesian Game**.
- **The players** are $N = \{1, 2\}$.
- **The types** are the cost functions of the companies $T_1 = \{c_1\} \quad T_2 = \{c_h, c_l\}$.
- **The utility functions** are the profits,

$$\begin{aligned}
\pi_1(q_1, q_2, c_1) &= (a - q_1 - q_2)q_1 - c_1 q_1 = (a - q_1 - q_2 - c_1)q_1 \\
\pi_2(q_1, q_h, c_h) &= (a - q_1 - q_h)q_h - c_h q_h = (a - q_1 - q_2 - c_h)q_h \\
\pi_2(q_1, q_l, c_l) &= (a - q_1 - q_l)q_l - c_l q_l = (a - q_1 - q_2 - c_l)q_l
\end{aligned}$$

- The **sets of actions** are $A_{c_1} = A_h = A_l = [0, \infty)$.
- **The conjectures of the firms** are

$$p_2(t_1 = c_1 | c_h) = p_2(t_1 = c_1 | c_l) = 1$$

and

$$p_1(t_2 = c_h | c_1) = \theta, \quad p_1(t_2 = c_l | c_1) = 1 - \theta$$

Best response of Firm 2

- If the cost function of Firm 2 is c_h then it solves the following problem

$$\max_{q_h} (a - q_1 - q_h) q_h - c_h q_h = \max_{q_2} (a - q_1 - q_h - c_h) q_h$$

- The FOC's are $a - q_1 - 2q_h - c_h = 0$ so the best response for Firm 2 provided its cost is c_h and that Firm 1 produces q_1 is

$$q_h = q_2(c_h) = \frac{a - q_1 - c_h}{2}$$

- If the cost function of Firm 2 is c_l then it solves the following problem

$$q_l = \max_{q_2} (a - q_1 - q_l) q_l - c_l q_l = \max_{q_2} (a - q_1 - q_l - c_l) q_l$$

- The FOC's for the firm are $a - q_1 - 2q_l - c_l = 0$ so the best response for Firm 2 if its cost is c_l and firm 1 produces q_1 is

$$q_l = q_2(c_l) = \frac{a - q_1 - c_l}{2}$$

Best response of Firm 1

- Firm 1 does not know the cost function of Firm 2. It maximizes **expected profit (EP)**

$$\max_{q_1} \quad \theta \underbrace{(a - q_1 - q_2(c_h) - c_1) q_1}_{\text{EP if firm 2's cost is } c_h} + (1 - \theta) \underbrace{(a - q_1 - q_2(c_l) - c_1) q_1}_{\text{EP if firm 2's cost is } c_l}$$

- The FOC is

$$\theta (a - 2q_1 - q_2(c_h) - c_1) + (1 - \theta) (a - 2q_1 - q_2(c_l) - c_1) = 0$$

- We obtain the reaction function of Firm 1

$$q_1 = \frac{\theta (a - q_2(c_h) - c) + (1 - \theta) (a - q_2(c_l) - c_1)}{2}$$

- The NE satisfies the equations

$$q_1 = \frac{\theta (a - q_2(c_h) - c) + (1 - \theta) (a - q_2(c_l) - c)}{2}$$

$$q_2(c_h) = \frac{a - q_1 - c_h}{2}$$

$$q_2(c_l) = \frac{a - q_1 - c_l}{2}$$

Solving for $q_2(c_h)$ and $q_2(c_l)$ in the first equation, we obtain

$$2q_1 = \theta \left(a - \frac{a - q_1 - c_h}{2} - c_1 \right) + (1 - \theta) \left(a - \frac{a - q_1 - c_l}{2} - c_1 \right)$$

- and from here we see that
$$q_1 = \frac{a - 2c_1 + \theta c_h + (1-\theta)c_l}{3}$$

- Substituting this value we obtain that

$$q_1^* = \frac{a - 2c_1 + \theta c_h + (1-\theta)c_l}{3}$$
$$q_2^*(c_h) = \frac{a - 2c_h + c_1}{3} + \frac{1-\theta}{6}(c_h - c_l)$$
$$q_2^*(c_l) = \frac{a - 2c_l + c_1}{3} - \frac{\theta}{6}(c_h - c_l)$$

Comparison with the Cournot equilibrium with complete information

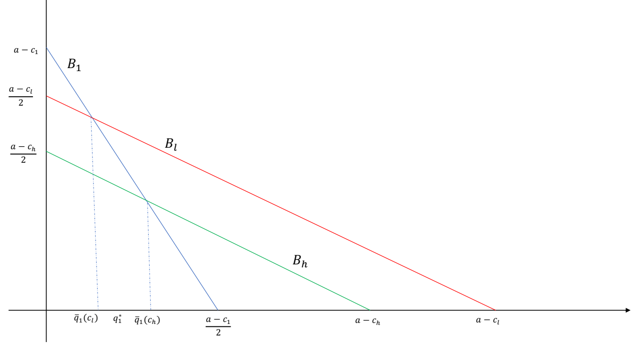
- With complete information and cost functions c_1, c_2 Cournot's equilibrium is
$$\bar{q}_1 = \frac{a - 2c_1 + c_2}{3} \quad \bar{q}_2 = \frac{a - 2c_2 + c_1}{3}$$

- We observe that the case with complete information can be obtained from the incomplete information case by setting $c_2 = c_h = c_l$.

- Let us call
$$\bar{q}_1(c_h) = \frac{a - 2c_1 + c_h}{3}$$
the production of Firm 1 if it knows that the cost function of Firm 2 is c_h.

- and
$$\bar{q}_1(c_l) = \frac{a - 2c_1 + c_l}{3}$$
the production of Firm 1 if it knows that the cost function of Firm 2 es c_l.

- Then,
$$q_1^* = \theta \bar{q}_1(c_h) + (1-\theta) \geq \bar{q}_1(c_l)$$
So,
$$\bar{q}_1(c_h) \geq q_1^* \geq \bar{q}_1(c_l)$$
Graphically,



That is, with asymmetric information firm 1 is

- better off than facing a firm with cost c_l and symmetric information;
- worse off than facing a firm with cost c_h and symmetric information,

In the presence of asymmetric information, firm 1 would benefit it were common knowledge that $c_2 = c_h$ and would be worse off $c_2 = c_l$ became common knowledge.

- Note that

$$\begin{aligned} q_2^*(c_h) &> \frac{a - 2c_h + c_1}{3} \\ q_2^*(c_l) &< \frac{a - 2c_l + c_1}{3} \end{aligned}$$

- Firm $c_2 = c_l$ benefits from asymmetric information. But, $c_2 = c_h$ is worse off with asymmetric information. With asymmetric information, firm $c_2 = c_l$ would benefit it could credibly reveal its type. However, in the presence of asymmetric information, firm $c_2 = c_h$ will want to hide its type and pretend that its cost is low.

- This happens because, under uncertainty, Firm 2 adjusts its production not only to its own cost, but it also takes into account that Firm 1 doesn't know the cost function of Firm 2 and adopts a production in between those that would adopt if it knew that the cost of Firm 2 were either h or l.

3 Applications.

3.1 Private Value Auctions.

- n agents participate in an auction. The agents bid simultaneously.

- Private value auctions: Each bidder i knows only his valuation of the object, $v_i \in [0, w]$, but not the valuation v_j of the other bidders. He only knows the distribution function F_j of v_j. Let us assume that bidders are symmetric. That is, the valuations of the bidders are independent and the same for all the bidders $F = F_1 = \cdots = F_n$.

- Let $f = F'$.

- We examine two possible types of auction.
  - **First-price sealed-bid auction:** The highest bidder wins: gets the object and pays his bid.
  - **Second-price sealed-bid auction:** The highest bidder gets the object and pays the second highest bid.

- In both cases, in case of a tie, the winner is determined by a fair lottery.

- Agents are risk neutral: If his valuation is v, wins the object and pays p, his payoff is $v - p$.

- A strategy for a bidder is a function $b_i : [0, w] \to [0, w]$,which determines his bid $b_i(v)$ for any value of v.

- We focus on symmetric equilibria equilibria: $b = b_1 = \cdots = b_n$.

- We answer the following questions:
  - What are symmetric equilibrium strategies in a first-price auction (1) and a second-price auction (2)?
  - From the point of view of the seller, which of the two auction formats yields a higher expected selling price in equilibrium?

- For concreteness, we focus on player 1.
- Let

$$Y_1(v) = \max\{b_2(v), \dots, b_n(v)\}$$

- Let G denote the distribution function of Y_1. We have that

$$G(y) = F(y)^{n-1}$$

and the density function is $g = G'$.

- We first write this mechanism as a Bayesian Game. For each $i = 1, \dots, n$,

1. The type of agent i is $T_i = [0, w]$.
2. The beliefs of agent i are described by $\text{prob}(v_i \leq x) = F(x)$.
3. A strategy for a bidder i is a function $b_i : [0, w] \to [0, w]$,which determines his bid $b_i(v)$ for any value of v.
4. The utility function of agent $i = 1$, given that
   - his valuation is v_1,
   - he bids b_1, and
   - the other agent bids b_j, $j = 2, \dots, n$

   is

$$u_1(b_1, \dots, b_n | v_1) = \begin{cases} v_1 - b_1, & \text{if } b_1 > Y_1; \\ 0, & \text{if } b_1 < Y_1; \\ (v_1 - b_1)/n & \text{if } b_i = b_j. \end{cases}$$

Second-price sealed-bid auctions

Proposition 3.1. *In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to* $b(x) = x$.

- Note that the above Proposition dependes only on the assumption of private values.
- How much each bidder expects to pay in equilibrium in a second-price auction?
- In a second-price auction, given that $b(x) = x$, the expected payment by bidder 1 with value x can be written as

$$\begin{aligned} m^2(x) &= \text{prob}\,[1 \text{ wins}] \times E\,[\text{2nd highest bid} | b_1(x) \text{ is the highest bid}] \\ &= \text{prob}\,[1 \text{ wins}] \times E\,[\text{2nd highest value} | x \text{ is the highest bid}] \\ &= G(x) \times E\,[Y_1 | Y_1 < x] \end{aligned}$$

First-price sealed-bid auctions

- $b(v) = v$ gives a payoff of 0. But bidding $b(v) < v$ gives a positive expected payoff. Thus, $b(v) = v$ is not a symmetric BNE.
- Suppose that bidders $j > 1$ follow a symmetric, increasing, and differentiable equilibrium strategy $\beta = \beta_2 = \cdots, = \beta_n$. Note $\beta(0) = 0$.
- Suppose bidder 1's type is x What is the best reply, $b_1 = \beta_1(x) = \text{BR}_1(\beta_2, \dots, \beta_n)$, of player 1?

- The best reply of agent 1 is the solution of the following maximization problem

$$\max_{b_1} \quad (v_1 - b_1)\text{prob}(b_1 > \beta(Y_1)) + \frac{1}{2}(v_1 - b_1)\text{prob}(b_1 = \beta(Y_1))$$

  Since, $\text{prob}(b_1 = \beta(Y_1)) = 0$ the above problem is equivalent to the following one

$$\max_{b_1}(v_1 - b_1)\text{prob}(b_1 > \beta(Y_1))$$

- We have $\text{prob}(b_1 > \beta(Y_1)) = \text{prob}(b_1 > \beta(Y_1)) = \text{prob}(Y_1 < \beta^{-1}(b_1)) = G\left(\beta^{-1}(b_1)\right)$.
- Agent's 1 maximization problem is

$$\max_{b_1}(x - b_1)G\left(\beta^{-1}(b_1)\right)$$

- The FOC is

$$(x - b_1)g\left(\beta^{-1}(b_1)\right)\frac{1}{\beta'\left(\beta^{-1}(b_1)\right)} = G\left(x\right)$$

- That is,

$$(x - \beta_1(x))g\left(x\right)\frac{1}{\beta'(x)} = G\left(x\right)$$

- Or,

$$(\beta_1(x)G\left(x\right))' = \beta_1'(x)G\left(x\right) + \beta_1(x)g\left(x\right) = xg(x)$$

- Hence,

$$\beta_1(x)G\left(x\right) = \int_0^x yg(y)\,dy$$

- We conclude

$$\beta_1(x) = \frac{1}{G(x)}\int_0^x yg(y)\,dy = E\left[Y_1|Y_1 < x\right]$$

- Note

$$\beta_1(x) = \frac{1}{G(x)}\int_0^x yg(y)\,dy = x - \frac{1}{G(x)}\int_0^x G(y)\,dy < x$$

  This is called bid 'shading'.
- Remark that

$$\beta_1(x) = x - \int_0^x \frac{G(y)}{G(x)}\,dy = x - \int_0^x \left(\frac{F(y)}{F(x)}\right)^{n-1}\,dy$$

  approaches x as n increases, because $F(y) < F(x)$, for $y < x$.

Example: Uniform distribution

- Let $w = 1$, $F(x) = x$. We have

$$\beta(x) = x - \frac{1}{x^{n-1}}\int_0^x y^{n-1}\,dy = \frac{n-1}{n}x$$

Example: Exponential distribution

- Let $w = \infty$, $F(x) = 1 - e^{-\lambda x}$, $n = 2$. We have

$$\beta(x) = x - \frac{1}{1 - e^{-\lambda x}} \int_0^x \left(1 - e^{-\lambda x}\right), dy = \frac{1}{\lambda} - \frac{x}{e^{\lambda x} - 1}$$

- The expected payment is

$$\text{prob}\,[1 \text{ wins}] \times \beta_1(x) = \text{prob}\,[1 \text{ wins}] \times E\left[Y_1 | Y_1 < x\right]$$

  which is the same as in the second price auction. This is a bit surprising.

- The expected revenues in the two auctions are the same.
  The ex ante expected payment of a particular bidder in either auction is

-

$$\begin{aligned} \int_0^w G(x) E\left[Y_1 | Y_1 < x\right] dx &= \int_0^w \left(\int_0^x y g(y)\, dy\right) f(x)\, dx = \int_0^w \left(\int_y^w f(x)\, dx\right) y g(y)\, dy \\ &= \int_0^w y\left(1 - F(y)\right) g(y)\, dy \end{aligned}$$

- The expected revenue of the seller is

$$n \int_0^w y\left(1 - F(y)\right) g(y)\, dy$$

- Note that the density function of the second highest payment is $n\left(1 - F(y)\right) g(y)$.
- Therefore, the expected revenue of the seller is the expectation of the second-highest value.

3.2 Contagion.

3.3 More information could be worse

- Consider the following game with incomplete information,

	L	M	R
T	**1,1/2**	**1,0**	**1,3/4**
B	**2,2**	**0,0**	**0,3**

	L	M	R
T	**1,1/2**	**1,3/4**	**1,0**
B	**2,2**	**0,3**	**0,0**

1 **2** w_1 1/2 w_2 1/2

- The agents believe that with probability 1/2 they play the game w_1 and with probability 1/2 they play the game w_2.
- The unique NE is (B, L). Each player gets an expected payoff of 2.
- Suppse now that player 2 knows if the true game played is w_1 or w_2. In this case,the unique NE is (T, R) or (T, M). The payoff of player 2 is $3/4$. He would have preferred not to be informed.
- But, can player 2 credibly commit to ignore the information he has received?