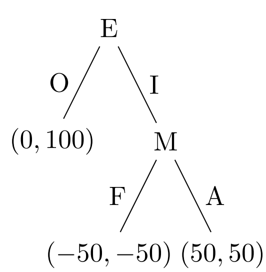
Chapter 3
Extensive Form Games

November 26, 2024

1 Extensive form games with Perfect Information

Extensive form (or sequential) games have a similar structure to normal form games. That is, they still consist of a set of players, their sets of strategies and payoffs. However, in an extensive form game **players take their decision sequentially**. In a **perfect information** extensive form game when players take their decisions they know the decisions taken by all the players in the past. Otherwise, the game is said to be an extensive form game with **imperfect information**.

Example 1.1. A monopolist (M) in a market which generates profits of $100 million faces the possibility of a competitor (E) entering the market. First the competitor decides whether to enter (I) or not (O) the market. If the competitor does enters the market, the monopolist observes the action of the entrant and decides whether to fight (F) or not (A) the entrant. If the monopolist fights it starts a price war. As a result both firms end up losing $50 million each. If the monopolist accommodates both firms share equally the market. Assume there is no cost of entry for the competitor. We may represent this situation as a tree.



In the previous example we see that the tree contains all the relevant information of the situation. Namely,

1. The players.
2. The possible sequential decisions of the players.
3. The information known by the players.
4. The payoffs of the players.

Definition 1.2 (Directed Tree)**.** For the purposes of this course, a directed graph consists of a finite set of vertices (or nodes) V and a set of edges E constructed as follows

- There is an initial node $x_0 \in V$.
- From the **initial node** x_0 there are edges that join the initial vertex x_0 with a subset $V_1 \subset V \setminus \{x_0\}$ of the set nodes. If $V_1 \cup \{x_0\} = V$ we finish here.

- Suppose $V_1 \neq V$. For each of the nodes in V_1 there are a set of edges (possibly empty) that connect that node with a a subset $V_2 \subset V \setminus (V_1 \cup \{x_0\})$ of the set of nodes not used previously. No node of V_2 is connected to two different nodes of V_1. If $V_1 \cup V_2 \cup \{x_0\} = V$ we finish here.

  Now we repeat the following step $i = 2, \ldots, n$ (a finite number of times).

- Suppose $V_i \neq V$. For each of the nodes in V_i there are a set of edges (possibly empty) that connect that node with a a subset $V_{i+1} \subset V \setminus (V_1 \cup \cdots \cup V_i \cup \{x_0\})$ of the set of nodes not used previously. No node of V_{i+1} is connected to two different nodes of V_i. If $V \setminus (V_1 \cup \cdots \cup V_{i+1} \cup \{x_0\} = V$ we finish here.

- The process ends when all the nodes (except the initial node) have been connected to a predecessor.

Every edge $a \in E$ goes from a vertex $x \in V$ into another vertex $y \in V$. We write $y = n(x, a)$ and we say x is the **predecessor** of y and y is one of the **successors** of x. We The set $A(x) \subset E$ is the set of edges that go out from the vertex x to other nodes in the tree.

The set of nodes Z that have no successor are called the **terminal nodes**.

Definition 1.3 (An extensive form game with perfect information)**.** An **extensive form game with perfect information** consists of a directed tree plus the following structure.

1. A finite set of players $N = \{1, 2, \ldots, I\}$.
2. A set of functions that describe for each $x \in V \setminus Z$ the player $i(x) \in N$ who moves at x. Each of the nodes has only one player associated to it.
3. The set of edges $A(x) \subset E$ is the set of possible actions of agent $i(x) \in N$ at the node $x \in V$.
4. The payoff functions $u_j : Z \to \mathbb{R}$, $j \in N$ which assigns payoffs to the players as a function of the terminal nodes.

Observation 1.4. To every extensive form game with perfect information we can associate a normal form game.

Definition 1.5 (NE of an extensive form game with perfect information)**.** The set of NE of an extensive form game with perfect information is the set of NE of the associated normal form game.

Example 1.6. Continuing with example 1.1, the set of strategies for the players are $S_1 = \{O, I\}$, $S_2 = \{F, A\}$. The game may be represented by the table

		Monopolist	
		A	F
Entrant	I	$50, 50$	$-50, -50$
	O	$0, 100$	$0, 100$

Let us look for a NE of the form

$$\begin{aligned} \sigma_1 &= xI + (1-x)O \\ \sigma_2 &= yA + (1-y)F \end{aligned}$$

We compute the expected utilities of the players

$$\begin{aligned} u_1(I, \sigma_2) &= 50y - 50(1-y) = 100y - 50 \\ u_1(O, \sigma_2) &= 0y + 0(1-y) = 0 \\ u_2(\sigma_1, A) &= 50x + 100(1-x) = 100 - 50x \\ u_2(\sigma_1, F) &= -50x + 100(1-x) = 100 - 150x \end{aligned}$$

Note that

(a) $0 < 100y - 50$ for $\frac{1}{2} < y \leq 1$.

(b) $0 > 100y - 50$ for $0 < y < \frac{1}{2}$.

(c) $0 = 100y - 50$ for $y = \frac{1}{2}$.

Thus, we have that best reply of player 1 is

$$\text{BR}_1(\sigma_2) = \begin{cases} x = 0 & \text{if} \quad 0 \leq y < \frac{1}{2} \\ [0,1] & \text{if} \quad y = \frac{1}{2} \\ x = 1 & \text{if} \quad \frac{1}{2} < y \leq 1 \end{cases}$$

Note now that

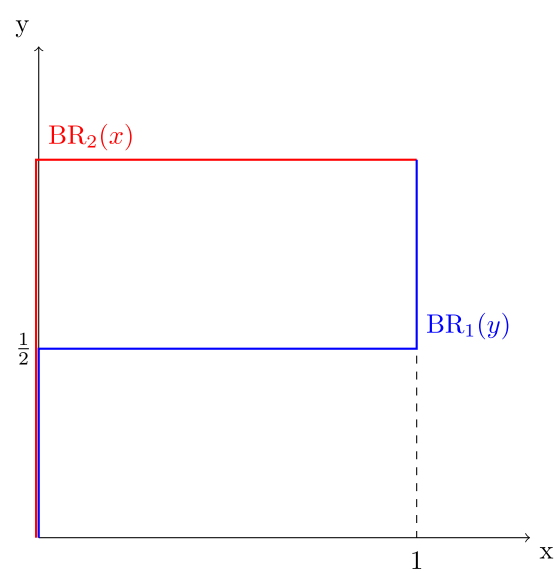
(a) $100 - 150x < 100 - 50x$ for $x > 0$.

(b) $100 - 150x = 100 - 50x$ for $x = 0$.

Thus, we have that best reply of player 2 is

$$\text{BR}_2(\sigma_1) = \begin{cases} y = 1 & \text{if} \quad 0 < x \leq 1 \\ [0,1] & \text{if} \quad x = 0 \end{cases}$$

Graphically,

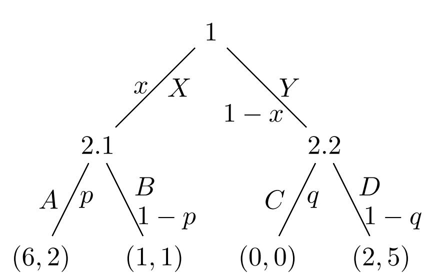


We see that there are two NE, (I, A) and (O, F), in pure strategies and the following NE

$$(O, yA + (1-y)F) \quad 0 \leq y \leq \frac{1}{2}$$

in mixed strategies

Example 1.7. Consider the following game



The set of strategies for the players are $S_1 = \{X, Y\}$, $S_2 = \{AC, AD, BC, BD\}$. The game may be represented by the table

		Player 2			
		$\overset{pq}{AC}$	$\overset{p(1-q)}{AD}$	$\overset{(1-p)q}{BC}$	$\overset{(1-p)(1-q)}{BD}$
Player 1	x X	$6, 2$	$6, 2$	$1, 1$	$1, 1$
	y Y	$0, 0$	$2, 5$	$0, 0$	$2, 5$

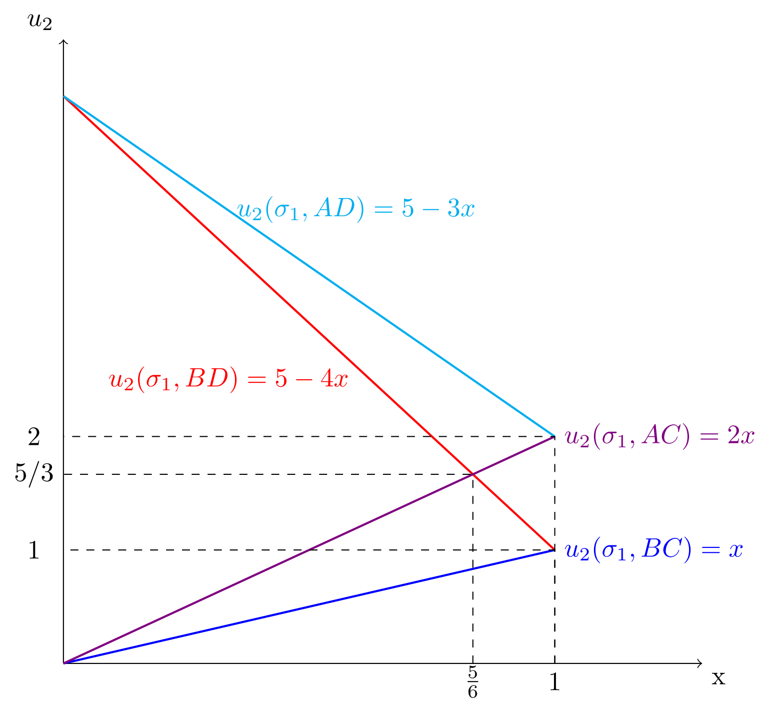
We look for a NE in mixed **behavioral** strategies of the form

$$\begin{aligned}\sigma_1 &= xX + (1-x)Y \\ \sigma_2 &= (pA + (1-p)B, qC + (1-q)D)\end{aligned}$$

The expected utilities of the players are

$$\begin{aligned}u_1(X, \sigma_2) &= 6pq + 6p(1-q) + (1-p)q + (1-p)(1-q) = 6p + 1 - p = 1 + 5p \\ u_1(Y, \sigma_2) &= 0pq + 2p(1-q) + 0(1-p)q + 2(1-p)(1-q) = 2(1-q) = 2 - 2q \\ u_2(\sigma_1, AC) &= 2x \\ u_2(\sigma_1, AD) &= 5 - 3x \\ u_2(\sigma_1, BC) &= x \\ u_2(\sigma_1, BD) &= 5 - 4x\end{aligned}$$

We represent graphically the utilities of player 2.



We see that

$$\text{BR}_2(\sigma_1) = \begin{cases} \{AD, BD\} \quad (q = 0) & \text{if} \quad x = 0 \quad (\sigma_1 = Y) \\ AD \quad (p = 1, q = 0) & \text{if} \quad 0 < x < 1 \\ \{AC, AD\} \quad (p = 1) & \text{if} \quad x = 1 \quad (\sigma_1 = X) \end{cases}$$

In addition we have that

$$\begin{aligned}\text{BR}_1(AC) &= X, \quad \text{BR}_2(X) = \{AC, AD\} \\ \text{BR}_1(AD) &= X, \quad \text{BR}_2(X) = \{AC, AD\} \\ \text{BR}_1(BD) &= Y, \quad \text{BR}_2(Y) = \{AD, BD\}\end{aligned}$$

We obtain the NE in pure strategies,

- (X, AC), with payoffs $u_1(X, AC) = 6$, $u_2(X, AC) = 2$,
- (X, AD), with payoffs $u_1(X, AD) = 6$, $u_2(X, AD) = 2$,
- (Y, BD), with payoffs $u_1(Y, BD) = 2$, $u_2(Y, BD) = 5$

Let us now look for a NE in which player 1 uses the strategy $\sigma_1 = X$ and player 2 uses a mixed strategy $\sigma_2 = (A, qC + (1-q)D)$. That is $p = 1$. We have already seen that $\mathrm{BR}_2(X) = \{AD, AC\}$. Now we need that $\mathrm{BR}_1(\sigma_2) = X$. That is, we must have that $(1+5p)|_{p=1} \geq 2 - 2q$. That is $6 \geq 2 - 2q$. This holds for any $0 \leq q \leq 1$. Hence, we have a continuum of NE of the form,

$$\begin{aligned} \sigma_1 &= X \\ \sigma_2 &= (A, qC + (1-q)D) \quad 0 \leq q \leq 1 \end{aligned}$$

with payoffs $u_1(X, \sigma_2) = 6$, $u_2(X, \sigma_2) = 2$.

Let us now look for a NE in which player 1 uses the strategy $\sigma_1 = Y$ and player 2 uses a mixed strategy $\sigma_2 = (pA + (1-p)B, D)$. That is $q = 0$. We have already seen that $\mathrm{BR}_2(Y) = \{AD, BD\}$. Now we need that $\mathrm{BR}_1(\sigma_2) = Y$. That is, we must have that $1 + 5p \leq (2-2q)|_{q=0}$. That is $1 + 5p \leq 2$. This holds for any $0 \leq p \leq 1/5$. Hence, we have a continuum of NE of the form,

$$\begin{aligned} \sigma_1 &= Y \\ \sigma_2 &= (pA + (1-p)B, D) \quad 0 \leq p \leq \frac{1}{5} \end{aligned}$$

with payoffs $u_1(X, \sigma_2) = 2$, $u_2(X, \sigma_2) = 5$.

2 Subgame perfect equilibrium of a game with perfect information

Definition 2.1 (Subgame of an extensive form game with perfect information)**.** Consider an extensive form game with perfect information. At each node of the game (including the initial node) which is not a final node, it starts a subgame. A subgame consists of the new starting node and all the subsequent nodes respecting the structure of the original game, the same players and the same payoffs.

Definition 2.2 (Subgame Perfect Nash Equilibrium (SPNE))**.** A subgame perfect equilibrium of an extensive form game is a NE of the game which induces also a NE in of each of its subgames.

Example 2.3. Let us consider again example 1.7. There are three subgames which start at the nodes 1, 2.1 and 2.2. The only NE of the subgame that starts at node 2.1 is A. The only NE of the subgame that starts at node 2.2 is D. Hence the only SPNE of the whole is (X, AD).

Observation 2.4 (Backward Induction Procedure). Consider an extensive form game with perfect information. The backward induction procedure is an algorithm that determines a recommendation of action for each of the players, in such a way that they are optimal at every decision node

1 The backward induction procedure starts with the nodes preceding the last nodes. In each of these nodes the player who owns that node chooses the action(s) that gives him the highest payoff.

2 Now look at the nodes that precede the actions in the previous step. Each of the players who are to move on those nodes anticipates the decisions of the players in the subsequent nodes and chooses the action(s) which maximized his payoffs.

3 Step 2 is repeated until the initial node is reached.

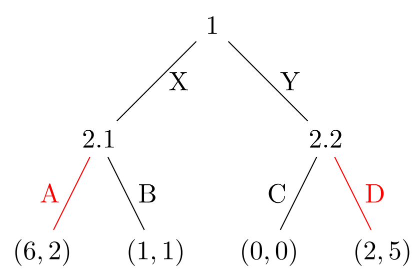
Example 2.5. Five Pirates have to decide how to distribute 100 gold coins they have captured from a ship. They do it according to the following rules.

- The pirates are numbered $1, 2, 3, 4, 5$.
- In round $i = 1, \ldots, 5$, pirate i proposes a distribution of the coins. If at least, half of the pirates vote for it, the proposal is accepted. Otherwise, the proposer is thrown into the water and drowns, and round $i + 1$ starts.
- For example, if there are two pirates left, the proposal will always win, and if there are four pirates, the proposer needs only one of the others voting for the proposal.

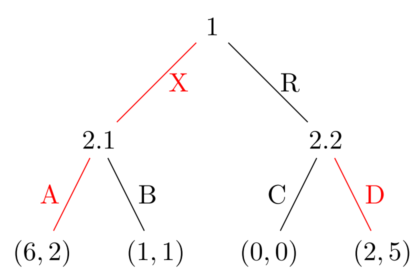
What is the SPNE?

Round	Pirate 1	Pirate 2	Pirate 3	Pirate 4	Pirate 5
1	98	0	1	0	1
2		99	0	1	0
3			99	0	1
4				100	0

Example 2.6. Let us apply the backward induction procedure to example 1.7. We start by considering the actions of player 2 at the nodes 2.1 and 2.2. The best action of player 2 at node 2.1 is A and the best action of player 2 at node 2.2 is D. Graphically,



Now player 1 anticipates these actions by player 2 and he should choose X at node 1.



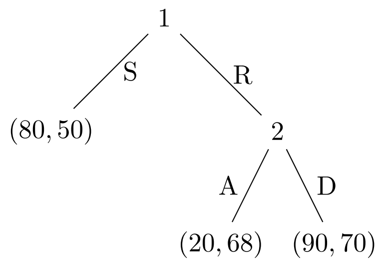
The backward induction procedure proposes the equilibrium $(X; AD)$.

Observation 2.7. For sequential games of perfect information the backward induction procedure and the SPNE select the same strategies for the players.

Observation 2.8. The SPNE eliminates the non-credible threats.

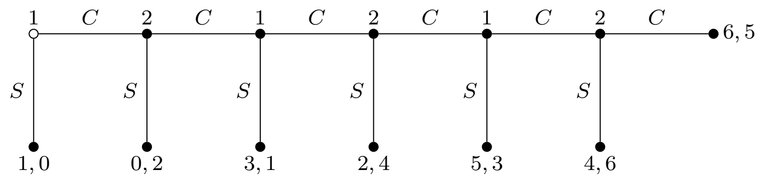
2.1 Limitations of SPNE

Example 2.9 (Trusting others to be rational)**.** Consider the following game by J. K. Goeree and C. A. Holt (AER, 2001).



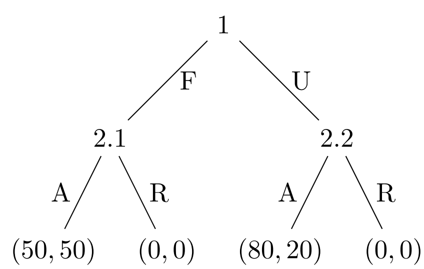
More than 50% of the players choose S.

Example 2.10 (The centipede game)**.** The following is a variant of the centipede game, first introduced by Robert Rosenthal in 1981.



The unique SPNE is for player 1 to choose S at node 1. However, on practice people relatively few players do so. Most players continue playing for a while (but not to very end) and achieve a higher payoff than the payoff predicted by the SPNE.

Example 2.11 (The ultimatum game)**.** The ultimatum game was first described by Werner GÃ¼th, Rolf Schmittberger, and Bernd Schwarze (1981). The following is a simple version of it.

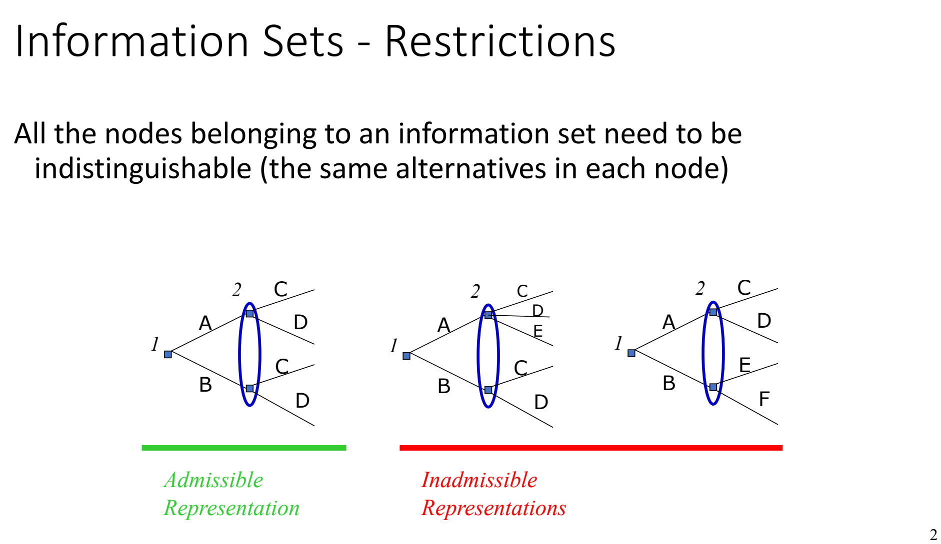


Experimentally, it has been found that people offer 'fair' (i.e., 50%–50%) splits. And offers of less than 30% of the total quantity to be shared are often rejected.

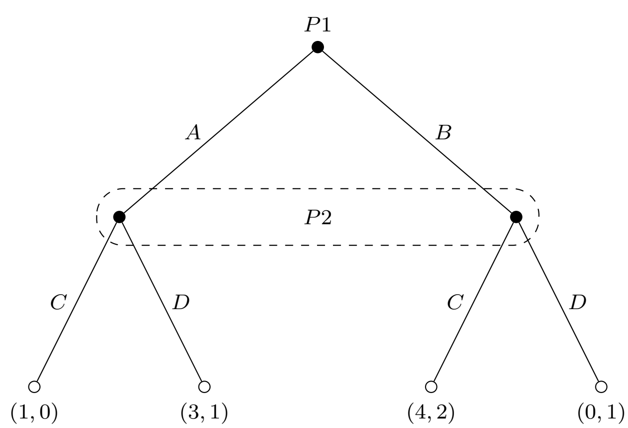
3 Extensive form games with Imperfect Information

Recall that in an extensive form game **imperfect information** there are some players who do not know some of the decisions taken by some other players. This is represented by **information sets.**

Definition 3.1 (Information Sets)**.** An information set is a set of nodes, all of which are associated to the same player. That player cannot distinguish among the vertices in that information set.



Example 3.2. Consider the following example.



In this example, when player 2 has to move, he does not know whether player 1 has chosen A or B. Note that this is equivalent to a one shot normal form game.

Observation 3.3. In a game of imperfect information,

1. If x and y are in the same information set, then $A(x) = A(y)$.
2. The player that owns an information takes the same action in all the nodes of that information set.

Observation 3.4. To every extensive form game with imperfect information we can associate a normal form game.

Definition 3.5 (NE of an extensive form game with imperfect information)**.** The set of NE of an extensive form game with imperfect information is the set of NE of the associated normal form game.

Example 3.6. Let us consider again example 3.2. The set of strategies for player 1 is $S_1 = \{A, B\}$ and the set of strategies for player 2 is $S_2 = \{C, D\}$. The normal form associated to this game is

		Player 2	
		C	D
Player 1	A	$1,0$	$3,1$
	B	$4,2$	$0,1$

Let us look for a NE of the form

$$\begin{aligned}\sigma_1 &= xA + (1-x)B \\ \sigma_2 &= yC + (1-y)D\end{aligned}$$

We compute the expected utilities of the players

$$\begin{aligned}u_1(A,\sigma_2) &= y + 3(1-y) = 3-2y \\ u_1(B,\sigma_2) &= 4y + 0(1-y) = 4y \\ u_2(\sigma_1,C) &= 0x + 2(1-x) = 2-2x \\ u_2(\sigma_1,D) &= 1x + 1(1-x) = 1\end{aligned}$$

Note that

(a) $3-2y > 4y$ for $y < \frac{1}{2}$.

(b) $3-2y = 4y$ for $y = \frac{1}{2}$.

(c) $3-2y < 4y$ for $y > \frac{1}{2}$.

Thus, we have that best reply of player 1 is

$$\text{BR}_1(\sigma_2) = \begin{cases} x=1 & \text{if} \quad 0 \leq y < \frac{1}{2} \\ [0,1] & \text{if} \quad y = \frac{1}{2} \\ x=0 & \text{if} \quad \frac{1}{2} < y \leq 1 \end{cases}$$

Note now that

(a) $2-2x > 1$ for $x < \frac{1}{2}$.

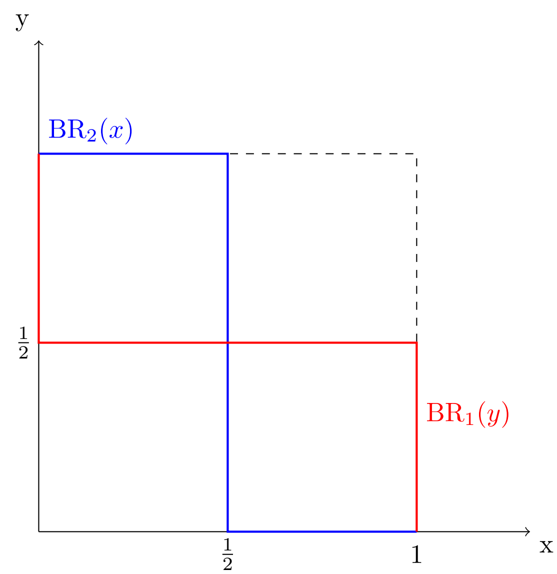
(b) $2-2x = 1$ for $x = \frac{1}{2}$.

(c) $2-2x < 1$ for $x > \frac{1}{2}$.

Thus, we have that best reply of player 2 is

$$\text{BR}_2(\sigma_1) = \begin{cases} y=1 & \text{if} \quad 0 < x < \frac{1}{2} \\ [0,1] & \text{if} \quad x = \frac{1}{2} \\ y=0 & \text{if} \quad \frac{1}{2} < x < 1 \end{cases}$$

Graphically,



We see that there are two NE, (B, C) and (A, D), in pure strategies and the following NE

$$\left(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}C + \frac{1}{2}D\right)$$

in mixed strategies

3.1 Subgame Perfect NE of an extensive form games with Imperfect Information

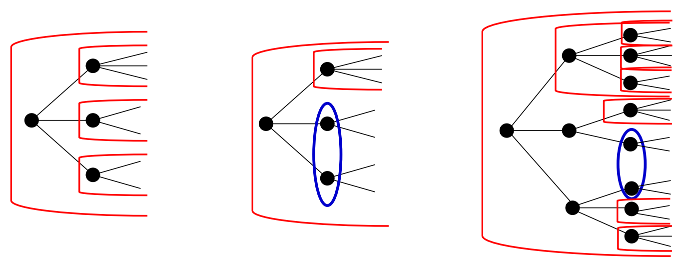
The backwards induction procedure cannot be applied directly to extensive form games with imperfect information. However, the notion of subgame perfect NE has a natural extension to extensive form games with imperfect information.

Definition 3.7 (Subgame of an extensive form game with imperfect information)**.** Consider an extensive form game with imperfect information. A subgame consists of a new starting node and all the subsequent nodes respecting the structure of the original game, the same players and the same payoffs. However, unlike in extensive form game with perfect information, in extensive form game with imperfect information a subgame may starts only at a node which fulfils the following two conditions:

1. The starting node may not be in an information set in which there are other nodes.
2. If the subgame contains a node that it is in the information set of some agent, then it contains also all the other nodes in that information set.

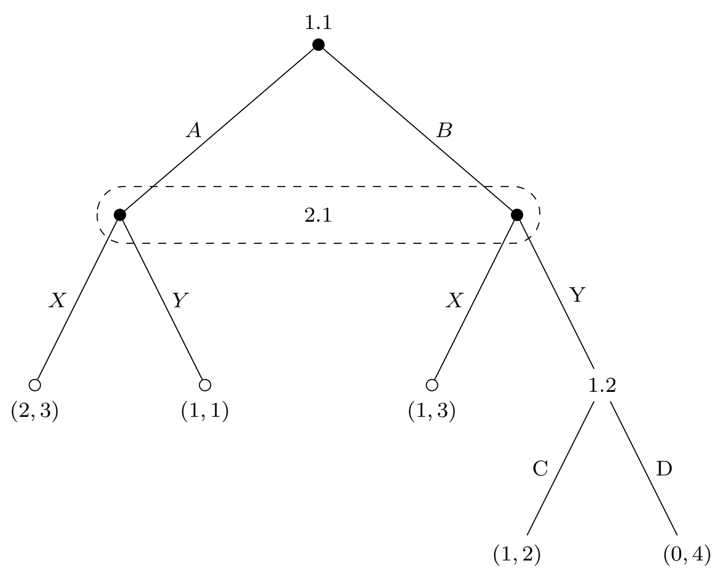
Subgames

All subgames of the following extensive form games:



Definition 3.8 (Subgame Perfect Nash Equilibrium (SPNE))**.** A subgame perfect equilibrium of an extensive form game is a NE of the game which induces also a NE in of each of its subgames.

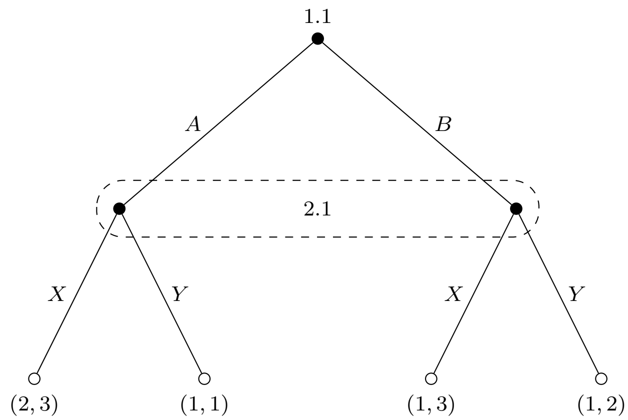
Example 3.9. Consider the following example



Let us compute the SPNE. The set of strategies for player 1 are $S_1 = \{AC, AD, BC, BD\}$. The set of strategies for player 2 are $S_2 = \{X, Y\}$. Hence a profile of strategies for the game is of the form $((1.1, 1.2), 2.1)$ or equivalently (α, β) where $\alpha \in \{AC, AD, BC, BD\}$ and $\beta \in \{X, Y\}$.

We find now the SPNE of the game. There are two subgames: 1.1 and 1.2.

The NE of the subgame that starts at 1.2 is C with payoffs $u_1 = 1$, $u_2 = 2$. We replace this subgame with by its payoffs in the whole game.



We compute the NE of this new game. The normal form game associated is the following.

Player 2

Player 1		X	Y
	A	$2,3$	$1,1$
	B	$1,3$	$1,2$

Strategy Y is dominated by X for player 2. We obtain the game

Player 2

Player 1		X
	A	$2,3$
	B	$1,3$

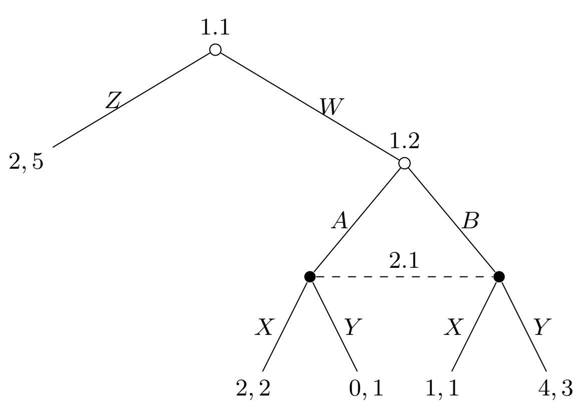
Now strategy B is dominated by A for player 1. We obtain the game

Player 2

Player 1		X
	A	$2,3$

Thus, the above game has a unique NE: (A, X). Now, the SPNE of the complete game is (AC, X) with payoffs $u_1 = 2$, $u_2 = 3$.

Example 3.10. Consider the following game.



Let us compute the SPNE. There are two subgames: (1.1) and (1.2). The normal form of the sub-game that starts at node 1.1 is

	X	Y
A	$\underline{2},\underline{2}$	$0,1$
B	$1,1$	$\underline{4},\underline{3}$

There are two NE in pure strategies: (A, X) with payoffs $(2, 2)$ and (B, Y) with payoffs $(4, 3)$. Now we look for a mixed strategy equilibrium of the form

$$\begin{aligned}\sigma_1 &= pA + (1-p)B \\ \sigma_2 &= qX + (1-q)Y\end{aligned}$$

The expected utilities of player 1 are

$$\begin{aligned}u_1(A, \sigma_2) &= 2q \\ u_1(B, \sigma_2) &= q + 4(1-q) = 4 - 3q\end{aligned}$$

So, player 1 is indifferent between the strategies X and Y if and only if $2q = 4 - 3q$, that is if and only if $q = \frac{4}{5}$. On the other hand, the expected utilities of player 2 are

$$\begin{aligned}u_2(\sigma_1, X) &= 2p + 1 - p = 1 + p \\ u_2(\sigma_1, Y) &= p + 3(1-q) = 3 - 2p\end{aligned}$$

So, player 2 is indifferent between the strategies A and B if and only if $1 + p = 3 - 2p$, that is if and only if $p = \frac{2}{3}$. We conclude that there is a mixed strategy NE of the form

$$\sigma = \left(\frac{2}{3}A + \frac{1}{3}B, \frac{4}{5}X + \frac{1}{5}Y\right)$$

with payoffs $u_1(\sigma) = \frac{8}{5}$ and $u_2(\sigma) = \frac{5}{3}$.

For each of the NE of the game that starts at 1.2 there is a corresponding SPNE of the complete game. We use the notation

$$((1.1, 1.2), 2.1)$$

to the denote the strategies followed by the players. We obtain the following NE.

- $((W, B), Y)$ with payoffs $u_1 = 4$, $u_2 = 3$.
- $\left(\left(Z, \frac{2}{3}A + \frac{1}{3}B\right), \frac{4}{5}X + \frac{1}{5}Y\right)$, with payoffs $u_1 = 2$, $u_2 = 5$.
- $((pZ + (1-p)W, A), X)$, $0 \leq p \leq 1$, with payoffs $u_1 = 2$, $u_2 = 5p + 2(1-p) = 2 + 3p$.