LAST NAME: NAME:

UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

MIDTERM EXAM–October 30th, 2024

Total time: 90 minutes. Calculators are allowed.

Exercise 1: Two firms, 1 and 2, produce complementary hardware and software products. This means that if one firm dedicates more resources to investment, it increases not only the demand of its own product but also the demand of the product of the other firm. Letting x_1 and x_2 denote the investment expenditures of 1 and 2, respectively, their profits are

$$
\begin{aligned}
u_1(x_1, x_2) &= (30 - x_2)x_1 + 40x_2 - \frac{x_1^2}{2} \\
u_2(x_1, x_2) &= (30 - x_1)x_2 + 40x_1 - \frac{x_2^2}{2}
\end{aligned}
$$

(a) **(20 points)** If firms 1 and 2 simultaneously decide their investment expenditures, what are the equilibrium investment expenditures? What are the profits of the firms?

Solution: *The best replies of the firms are*

$$
\begin{aligned}
\mathrm{BR}_1(x_2) &= 30 - x_1 \\
\mathrm{BR}_2(x_1) &= 30 - x_1
\end{aligned}
$$

The symmetric NE is $x_1^* = x_2^* = 15$. *And the profits are* $u_1^* = u_2^* = 1425/2 = 712.5$

(b) **(5 points)** Suppose now that firm 1 can make a take-it-or-leave-it offer to buy firm 2 from its owners. What is the lowest amount that the owners of firm 2 would be willing to accept?

Solution: *The lowest amount that the owners of firm* 2 *would be willing to accept is* 712.5.

(c) **(10 points)** Suppose that the owners of firm 2 accept the offer, and that the new firm will decide both x_1 and x_2. What are the optimal investment expenditures for the new firm? What profit will it obtain?

Solution: *The new firm will choose* x_1 *and* x_2 *so as to maximize* $u(x_1, x_2) = u_1(x_1, x_2) + u_2(x_1, x_2) = -\frac{x_1^2}{2} - 2x_1x_2 + 70x_1 - \frac{x_2^2}{2} + 70x_2$. *The solution is* $x_1^* = x_2^* = 70/3 \approx 23.33$. *The profit is* $= 4900/3 \approx 1633.33$.

(d) **(5 points)** Suppose that firm 1 makes the lowest offer that would be accepted by the owners of firm 2, that they accept such offer and that the new firm will then set the optimal investment expenditure levels that you have found in part (d). Would the profits of firm 1 (net of the price paid to buy firm 2) increase? If so, by how much?

Solution: *The increase would be* $4900/3 - 1425/2 - 1425/2 = 625/3 \approx 208.33$.

Exercise 2: Consider a Cornout duopoly which operates in a market with the following inverse demand function

$$P(Q) = \begin{cases} 180 - Q & \text{if } Q \leq 180, \\ 0 & \text{if } Q > 180. \end{cases}$$

where $Q = q_1 + q_2$ is the total output in the market. The cost of firm 2 is $c_2(q_2) = 12q_2$ with probability 1/2 and $c_2(q_2) = 24q_2$ with probability 1/2. The cost of firm 1 is $c_1(q_1) = 18q_1$. Firm 2 knows its own cost, but firm 1 only knows the types of costs of firm 2 and its probabilities. The above description is common knowledge.

(a) **(5 points)** Represent the above situation as a Bayesian Game. That is, describe the set set of players, their types, the set of strategies, their beliefs and their utilities.

Solution: *There are two players,* $N = \{1, 2\}$. *Their types are* $T_1 = \{r\}$, $T_2 = \{c_l, c_h\}$ *where* c_l *represents the situation when firm 2 knows that its marginal cost is* $c_l = 12$ *and* c_h *represents the situation when firm 2 knows that its marginal cost is* $c_h = 24$. *The sets of strategies are* $S_1 = [0, \infty)$ *and* $S_2 = [0, \infty) \times [0, \infty) = \{(s_l, s_h) : s_l, s_h \in [0, \infty)\}$. *Here* s_l *(resp.* s_h*) represents the strategy of firm 2 when it knows that its marginal cost is* $c_l = 12$ *(resp.* $c_h = 24$*). The beliefs of the players are*

$$p_1(c_l|r) = \frac{1}{2}; \quad p_1(c_h|r) = \frac{1}{2}, \quad p_2(r|a_l) = p_2(r|a_h) = 1$$

The utilities of the players are the following.

$$\begin{aligned} u_1(q_1, q_2|r) &= q_1(162 - q_1 - q_2) \\ u_2(q_1, q_2|c) &= q_2(180 - c - q_1 - q_2), \quad c = c_l, c_h \end{aligned}$$

(b) **(15 points)** Compute the Bayesian Equilibrium and the benefits of the firms in this equilibrium.

Solution: *The best response of type* c_l *firm 2 is the solution of the following maximization problem*

$$\max_{q_l} q_l(168 - q_1 - q_l)$$

The solution is

$$q_l = \frac{168 - q_1}{2} \tag{1}$$

The best response of type c_h *firm 2 is the solution of the following maximization problem*

$$\max_{q_h} q_h(156 - q_1 - q_h)$$

The solution is

$$q_h = \frac{156 - q_1}{2} \tag{2}$$

The best response of firm 1 is the solution of the following maximization problem

$$\max_{q_1} \frac{1}{2}\left(162 - q_1 - q_l\right) q_1 + \frac{1}{2}\left(162 - q_1 - q_h\right) q_1$$

The solution is

$$q_1 = \frac{324 - q_l - q_h}{4} \tag{3}$$

The Bayesian–Nash equilibrium is the solution to equations (1), (2) *and* (3). *We obtain,*

$$q_1 = 54, \quad q_l = 57, \quad q_h = 51$$

The benefits are,

$$\Pi_1 = 2916 \quad \Pi_l = 3249 \quad \Pi_h = 2601$$

(c) **(5 points)** Suppose now that firm 1 knows that the costs of firm 2 is $c_2(q_2) = 12q_2$. Compute the Nash equilibrium and the benefits of the firms in this equilibrium.

Solution: *The best response of type c_l firm 2 is the solution of the following maximization problem*

$$\max_{q_l} q_l(168 - q_1 - q_l)$$

The solution is

$$q_l = \frac{168 - q_1}{2} \tag{4}$$

The best response of firm 1 is the solution of the following maximization problem

$$\max_{q_1} \left(162 - q_1 - q_l\right) q_1$$

The solution is

$$q_1 = \frac{162 - q_l}{2} \tag{5}$$

The Bayesian–Nash equilibrium is the solution to equations (4) *and* (5). *We obtain,*

$$q_1 = 52, \quad q_l = 58$$

The benefits are,

$$\Pi_1 = 2704 \quad \Pi_l = 3364$$

(d) **(5 points)** Suppose now that firm 1 knows that the costs of firm 2 is $c_2(q_2) = 24q_2$. Compute the Nash equilibrium and the benefits of the firms in this equilibrium.

Solution: *The best response of type c_h firm 2 is the solution of the following maximization problem*

$$\max_{q_h} q_h(156 - q_1 - q_h)$$

The solution is

$$q_h = \frac{156 - q_1}{2} \tag{6}$$

The best response of firm 1 is

$$q_1 = \frac{162 - q_h}{2}$$

The Bayesian–Nash equilibrium is the solution to equations (4) *and* (6)*. We obtain,*

$$q_1 = 56, \quad q_h = 50$$

The benefits are,

$$\Pi_1 = 3136 \quad \Pi_h = 2500$$

(e) **(5 points)** Suppose that firm 1 can pay an amount so that the cost of firm 2 becomes public. What is the maximum amount firm 1 would be willing to pay to make that information public?

Solution: *The maximum value firm 1 would be willing to pay is*

$$\frac{1}{2} \times 2704 + \frac{1}{2} \times 3136 - 2916 = 4$$

(f) **(5 points)** Suppose that firm 1 can pay an amount so that it knows privately the cost of firm 2 and then it decides if it makes it public. What is the maximum amount firm 1 would be willing to pay to acquire that information privately?

Solution: *Now, if firm 1 finds out that the cost of firm 2 is low it will not inform firm 2 that has this information. In such a case it would produce* $\mathrm{BR}_1(57) = \frac{105}{2}$ *with a profit of* $u_1\left(\frac{105}{2}, 57\right) = \frac{11025}{4} \approx 2756.25$

If if firm 1 finds out that the cost of firm 2 is high and does not inform firm 2 that has this information, then $q_2 = 51$ *and* $\mathrm{BR}_1(51) = \frac{111}{2}$, $u_1\left(\frac{111}{2}, 51\right) = 12321/4 \approx 3080.25$. *Whereas if firm 1 informs firm 2 that it knows that* $c_h = 24$*, then* $q_2 = 50$, $\mathrm{BR}_1(50) = 56$, $u_1(56, 50) = 3136$. *Hence, if if firm 1 finds out that the cost of firm 2 is high it informs firm 2 that it knows its cost.*

The maximum value firm 1 would be willing to pay is

$$\frac{1}{2} \times \frac{11025}{4} + \frac{1}{2} \times 3136 - 2916 = \frac{241}{8} = 30.125$$

Exercise 3: Two students (the agents) start to share an apartment with a plant in the living room. Each of them, say agent $i = 1, 2$ can decide to water ($s_i = 1$) or not to water ($s_i = 0$) the plant at a cost c. If the plant is watered by at least one of them, then it looks beautiful. If the plant is not watered, it dies. The utility of agent $i = 1, 2$ when the plant is watered is $t_i^2 \in [0, 1]$. Thus the payoffs of the agents are

$$u_i = (s_1, s_2; t_i) = \max\{s_1, s_2\} t_i^2 - s_i c \quad i = 1, 2$$

They do not know each other well, so they are not sure how much the other student cares about the plant. Agent i knows t_i and knows that for $j \neq i$, t_j is distributed uniformly in $[0, 1]$.

(a) **(5 points)** Describe the situation as Bayesian game. That is write the set of agents, the set of actions, the set of types of the agents, their beliefs and the payoffs.

Solution: *The set of agents is* $N = \{1,2\}$. *The set of actions are* $A_1 = A_2 = \{0,1\}$. *The sets of types of the agents are* $T_1 = T_2 = [0,1]$. *A strategy for agent* $i = 1,2$ *is a mapping* $s_i : A_i \to T_i$. *The beliefs of the agents are*

$$p_i\left(t_j \leq x | t_i\right) = x \quad i,j = 1,2 \quad i \neq j$$

(b) **(15 points)** Prove that there is a Bayesian Nash Equilibrium of the form

$$s_i\left(t_i\right) = \begin{cases} 0, & \text{for } t_i < a \\ 1, & \text{for } t_i \geq a \end{cases} \qquad i = 1,2$$

for some $a \in [0,1]$. You have to determine the value of a as part of your solution.

Solution:

Assume that player 2 follows the strategy

$$s_2\left(t_2\right) = \begin{cases} 0, & \textit{for } t_2 < a \\ 1, & \textit{for } t_2 \geq a \end{cases}$$

for some a appropriately chosen. And let us compute the best reply of player 1. The (expected) utility of player 1 when he chooses $s_1 = 1$ *is*

$$u_1\left(1, s_2; t_1\right) = t_1^2 - c$$

And the (expected) utility of player 1 when he chooses $s_1 = 0$ *is*

$$u_1\left(0, s_2; t_1\right) = t_1^2 \times \text{prob}\left(t_2 \geq a\right) = t_1^2(1-a)$$

Note that $u_1(0,0) = 0$. *Let* $\varepsilon > 0$. *We want that if* $t_1 = a + \varepsilon$ *then* $\text{BR}(s_2; t_1) = 0$ *and if* $t_1 = a - \varepsilon$ *then* $\text{BR}(s_2; t_1) = 1$. *That is we want that*

$$(a+\varepsilon)^2 - c \geq (a+\varepsilon)^2(1-a) = (a+\varepsilon)^2 + a(a+\varepsilon)^2$$
$$(a-\varepsilon)^2 - c \leq (a-\varepsilon)^2(1-a) = (a-\varepsilon)^2 + a(a-\varepsilon)^2$$

That is,

$$a(a+\varepsilon)^2 \geq c$$
$$a(a-\varepsilon)^2 \leq c$$

for every $\varepsilon > 0$. *This is possible only if* $a^3 = c$ *or* $a = c^{1/3}$. *We have proved that if*

$$s_2\left(t_2\right) = \begin{cases} 0, & \textit{for } t_2 < c^{1/3} \\ 1, & \textit{for } t_2 \geq c^{1/3} \end{cases}$$

then,

$$s_1\left(t_1\right) = \begin{cases} 0, & \textit{for } t_1 < c^{1/3} \\ 1, & \textit{for } t_1 \geq c^{1/3} \end{cases}$$

is the best reply of player 1. Similarly, we can prove that the best reply of player 1 to s_1 *above is* s_2.