LAST NAME: NAME:

UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

MIDTERM EXAM–November 2nd, 2023

Total time: 90 minutes. This is an open book exam. Calculators are allowed.

Exercise 1: Assume that two firms, 1 and 2, produce heterogeneous products and the quantities demanded by the market, when these firms fix prices p_1 and p_2, are, respectively:

$$\begin{aligned} x_1(p_1,p_2) &= 100 - p_1 + \frac{p_2}{2} \\ x_2(p_1,p_2) &= 100 - p_2 + \frac{p_1}{3} \end{aligned}$$

These demand functions describe a situation in which products are not perfectly homogenous. Suppose that both firms have constant marginal cost 38 and set prices simultaneously.

(a) **(10 points)** Write the profit functions $\pi_1(p_1,p_2)$ and $\pi_2(p_1,p_2)$.

Solution: *The profits are*

$$\begin{aligned} \Pi_1(p_1,p_2) &= x_1(p_1-c) = \left(100 - p_1 + \frac{p_2}{2}\right)(p_1-38) = 138p_1 - p_1^2 + \frac{p_2p_1}{2} - 19p_2 - 3800 \\ \Pi_2(p_1,p_2) &= x_2(p_2-c) = \left(100 - p_2 + \frac{p_1}{3}\right)(p_2-38) = 138p_2 - p_2^2 + \frac{p_1p_2}{3} + -\frac{38p_1}{3} - 3800 \end{aligned}$$

(b) **(10 points)** Compute the best reply functions $\mathrm{BR}_1(p_2)$ and $\mathrm{BR}_2(p_1)$ of the firms.

Solution: *The first order conditions are*

$$\begin{aligned} 0 &= -2p_1 + \frac{p_2}{2} + 138 \\ 0 &= \frac{p_1}{3} - 2p_2 + 138 \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{BR}_1(p_2) &= \frac{1}{4}(p_2 + 276) = \frac{p_2}{4} + 69 \\ \mathrm{BR}_2(p_1) &= \frac{1}{6}(p_1 + 414) = \frac{p_1}{6} + 69 \end{aligned}$$

Note that the second order conditions

$$\frac{\partial^2 \Pi_1}{\partial p_1 \partial p_1} = \frac{\partial^2 \Pi_2}{\partial p_2 \partial p_2} = -2$$

are fullfilled.

(c) **(10 points)** Find the Nash equilibrium (p_1^*, p_2^*) and the equilibrium profits, $\pi_1(p_1^*, p_2^*)$ and $\pi_2(p_1^*, p_2^*)$.

Solution:

The prices $p_1 = 0$ and $p_2 = 0$ give 0 profits. Hence, The NE is the solution of the system of equations

$$\begin{aligned} p_1 &= \frac{1}{4}(p_2 + 276) \\ p_2 &= \frac{1}{6}(p_1 + 414) \end{aligned}$$

The solution is $p_1^* = 90$, $p_2^* = 84$. The profits are $\pi_1(90,84) = 2704$ and $\pi_2(90,84) = 2116$.

(d) **(10 points)** Suppose now that firm 2 can launch an advertising campaign such that its new demand function is

$$x_2(p_1, p_2) = 100 - p_2 + \frac{p_1}{2}$$

How much would firm 2 be willing to pay for such an advertising campaign?
Suggestion: Note that $x_2(p_1, p_2)$ is now symmetric with respect firm 1. This simplifies greatly the computations.

Solution: *By symmetry the best reply of firm 2 is*

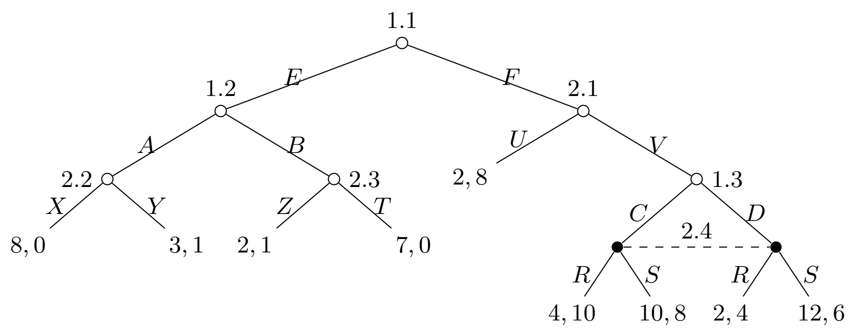
$$\mathrm{BR}_1(p_1) = \frac{1}{4}(p_1 + 276)$$

The equilibrium is symmetric $p = p_1 = p_2$,

$$p = \frac{1}{4}(p + 276)$$

We obtain $\bar{p}_1 = \bar{p}_2 = 92$. *The profits are* $\pi_1(92, 92) = \pi_2(92, 92) = 2916$. *Firm 2 is willing to pay up to* $2916 - 2116 = 800$ *for the advertising campaign.*

Exercise 2: Consider the following dynamic form game.



(a) **(20 points)** Write the normal form of the game that starts at node 1.3 and find all (that is, in pure and mixed strategies) NE of this game.

Solution: *The normal form game is*

	R	S
C	$4, 10$	$10, 8$
D	$2, 4$	$12, 6$

We see that there are two NE in pure strategies: (C, R) *with payoffs* $u_1 = 4$, $u_2 = 10$ *and* (D, S) *with payoffs* $u_1 = 12$, $u_2 = 6$.

Now we look for NE in mixed strategies of the form

$$\sigma_1 = pC + (1-p)D$$
$$\sigma_2 = qR + (1-q)S$$

The expected payoffs of player 1 are

$$u_1(C, \sigma_2) = 10 - 6q$$
$$u_1(D, \sigma_2) = 12 - 10q$$

Equating $10 - 6q = 12 - 10q$, *we obtain* $q = \frac{1}{2}$, $u_1(C, \sigma_2) = u_1(D, \sigma_2) = 7$.*The expected payoffs of player 2 are*

$$u_2(\sigma_1, R) = 4 + 6p$$
$$u_2(\sigma_2, S) = 6 + 2p$$

Equating $4 + 6p = 6 + 2p$, *we obtain* $p = \frac{1}{2}$, $u_2(\sigma_1, R) = u_2(\sigma_2, S) = 7$.

Thus, the NE of this game are:

(a) (C, R) *with payoffs* $u_1 = 4$, $u_2 = 10$ *;*

(b) (D, S) *with payoffs* $u_1 = 12$, $u_2 = 6$*; and*

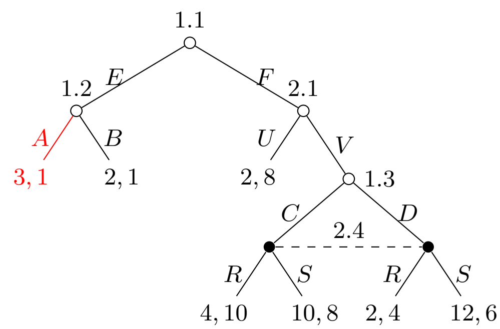
(c) $\left(\frac{1}{2}C + \frac{1}{2}D, \frac{1}{2}R + \frac{1}{2}S\right)$, *with payoffs* $u_1 = 7$, $u_2 = 7$.

(b) **(20 points)** Find the subgame perfect Nash equilibria of the complete game. Compute the utilities attained by the players in each SPNE.

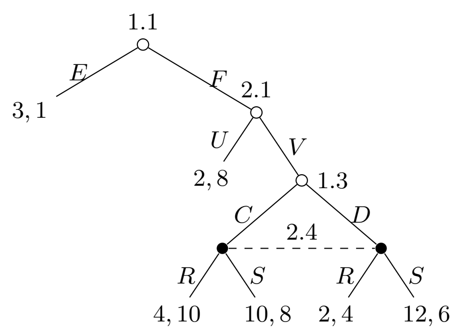
Solution: *We use the notation*

$$(1.1; (1.2, (2.2, 2.3)); (2.1, 1.3))$$

At node 2.2 *, player 2 chooses* Y*. The payoffs of the sub-game starting at node* 2.2 *are* $u_1 = 3$, $u_2 = 1$*. At node* 2.3 *, player 2 chooses* Z*. The payoffs of the sub-game starting at node* 2.2 *are* $u_1 = 2$, $u_2 = 1$.



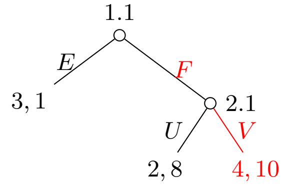
At node 1.2*, player 1 chooses* A.



All the SPNE are of the form

$$(1.1; (A, (Y, Z)); (2.1, 1.3))$$

(a) Let us look for a SPNE in which at node 1.3 *the NE* (C, R) *is played.*

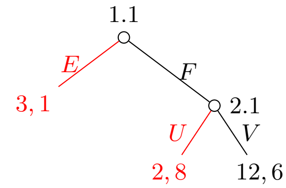


At node 2.1 *player 2 chooses* V*. At node* 1.1 *player 1 chooses* F*. We have the SPNE*

$$(F; (A, (Y, Z)); (V, (C, R)))$$

with payoffs $u_1 = 4$, $u_2 = 10$.

(b) Let us look for a SPNE in which at node 1.3 *the NE* (D, S) *is played.*

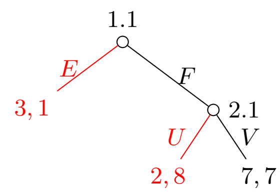


At node 2.1 *player 2 chooses* U*. At node* 1.1 *player 1 chooses* E*. We have the SPNE*

$$(E; (A, (Y, Z)); (U, (D, S)))$$

with payoffs $u_1 = 3$, $u_2 = 1$.

(c) Let us look for a SPNE in which at node 1.3 *the NE* $\left(\frac{1}{2}C + \frac{1}{2}D, \frac{1}{2}R + \frac{1}{2}S\right)$ *is played.*



At node 2.1 *player 2 chooses* U*. At node* 1.1 *player 1 chooses* E*. We have the SPNE*

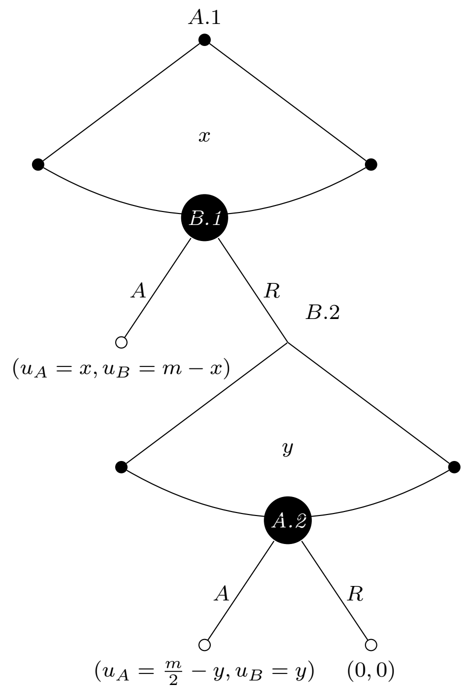
$$\left(E; (A, (Y, Z)); \left(\frac{1}{2}C + \frac{1}{2}D, \frac{1}{2}R + \frac{1}{2}S\right)\right)$$

with payoffs $u_1 = 3$, $u_2 = 1$.

Exercise 3: Two agents engage in a sequential bargaining process as follows:

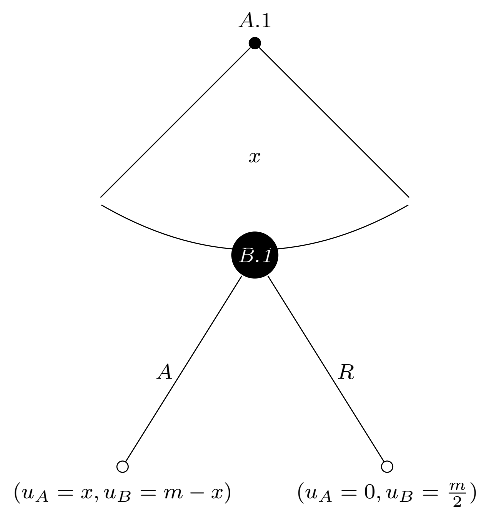
1. In **stage** 1, agent A proposes $0 \leq x \leq m$, with $m > 0$.
   (a) If agent B accepts, agent A gets x, agent B gets $m - x$.
   (b) If agent B refuses, they go into the stage 2.
2. In **stage** 2, agent B proposes $0 \leq y \leq \frac{m}{2}$.
   (a) If agent A accepts, agent A gets $\frac{m}{2} - y$, agent B gets y.
   (b) If agent A refuses, agent A gets the amount 0 and agent B gets the amount 0.

(a) **(5 points)** Describe the situation as an extensive form game. Draw the tree that represents the game.
Solution:



(b) **(10 points)** Compute the subgame perfect Nash equilibria of this game. Write the strategies of the players at each node. Write the payoffs of the players in each of the SPNE.

Solution: *At node A.2, agent A accepts iff $\frac{m}{2} - y \geq c$. That is agent A accepts iff $y \leq \frac{m}{2}$. Given the best reply of agent A at node A.2, the best response of agent B at node B.2 is to offer $y = \frac{m}{2}$. Thus, we may assume that if we ever reach node B.2, agent B will offer $y = \frac{m}{2}$ and agent A accepts. The payoffs will be $u_A = 0$, $u_B = \frac{m}{2} > 0$. We replace this payoffs at node A.2*



Now, player B at node B.1 accepts iff $m - x \geq \frac{m}{2}$. That, is at node B.1 player B accepts iff $x \leq \frac{m}{2}$. The best response now for player A is to offer $x = \frac{m}{2}$ at node A.1. The SPNE is the following.

- *Node A.1: $x = \frac{m}{2}$.*
- *Node B.1: accept iff $x \leq \frac{m}{2}$.*
- *Node B.2: $y = \frac{m}{2}$.*

- *Node A.2: accept iff $y \leq \frac{m}{2}$.*

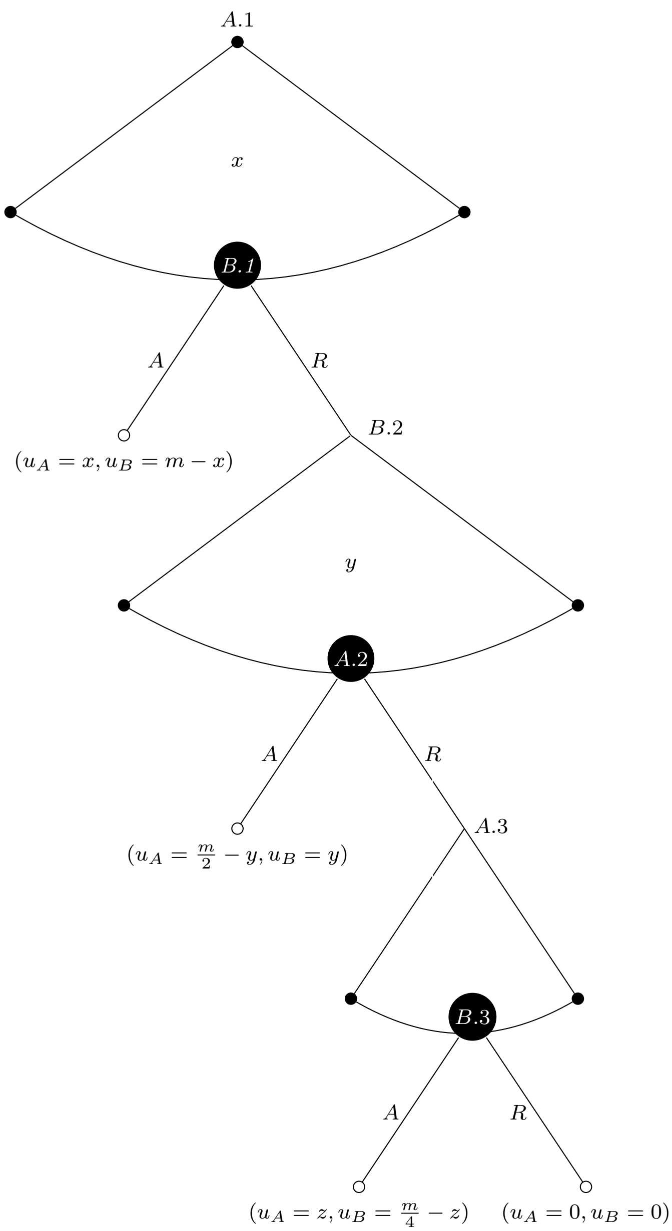
The payoffs are $u_A = \frac{m}{2}$, $u_B = \frac{m}{2}$.

(c) Imagine now that, if, in the second stage agent A refuses, they go into a third stage:

3. In **stage** 3, agent A proposes $0 \leq z \leq \frac{m}{4}$. If agent B accepts, agent A gets z, agent B gets $\frac{m}{4} - z$. If agent B refuses, agent A gets the amount 0 and agent B gets the amount 0.

(a) **(5 points)** Describe the new dynamic game

Solution:



(b) **(10 points)** Compute the SPNE. Write the strategies of the players at each node. Compute the payoffs in the SPNE.

Solution: *The SPNE is the following.*

- *Node A.1: $x = \frac{3m}{4}$.*

- *Node B.1: accept iff* $x =\leq \frac{3m}{4}$.
- *Node B.2:* $y = \frac{m}{4}$.
- *Node A.2: accept iff* $y \leq \frac{m}{4}$.
- *Node A.3:* $z = \frac{m}{4}$.
- *Node B.3: accept iff* $z \leq \frac{m}{4}$.

The payoffs are $u_A = \frac{3m}{4}$, $u_B = \frac{m}{4}$.

(c) **(10 points)** Who prefers the 2-stage bargaining procedure versus the 3-stage bargaining procedure?
Solution:
Agent A prefers the three-stage game and agent B prefers the two-stage game.