Name (Print): ______________________________

University Carlos III de Madrid

Master in Economics
Master in Industrial Economics and Markets

Final Exam. Game Theory. 01/08/2025.

Time Limit: 120 Minutes.

Exercise	Points	Score
1	35	
2	65	
3	65	
4	30	
Total:	195	

1. Consider the following normal form game:

	X	Y	Z	U
A	$2,0$	$0,2$	$4,9$	$10,8$
B	$3,15$	$1,11$	$2,6$	$5,5$
C	$2,20$	$0,5$	$1,7$	$2,6$

(a) (5 points) What are the strategies that survive the iterated elimination of strictly dominated strategies?

Solution: Strategy U is dominated by strategy Z for player 2. After eliminating this strategy we obtain the following game

	X	Y	Z
A	$2,0$	$0,2$	$4,9$
B	$3,15$	$1,11$	$2,6$
C	$2,20$	$0,5$	$1,7$

Now strategy C is dominated by strategy B for player 1. After eliminating this strategy we obtain the following game

	X	Y	Z
A	$2,0$	$0,2$	$4,9$
B	$3,15$	$1,11$	$2,6$

Note also that the mixed strategy $\sigma_2 = \frac{2}{3}X + \frac{1}{3}Z$ dominates Y, because

$$\begin{aligned} u_2(A, \sigma_2) &= \frac{2}{3} \times 0 + \frac{1}{3} \times 9 = 3 > 2 = u_2(A, Y) \\ u_2(C, \sigma_2) &= \frac{2}{3} \times 15 + \frac{1}{3} \times 6 = 12 > 11 = u_2(C, Y) \end{aligned}$$

(See also the argument in part (c) below).
The rationalizable strategies are $\{A, C\} \times \{X, Z\}$.

(b) (10 points) Find all pure strategy Nash equilibria and the payoffs of these equilibria.

Solution: *This is solved in part (c).*

(c) (20 points) Compute the mixed strategy Nash equilibria and the expected payoffs of these equilibria.

Solution:

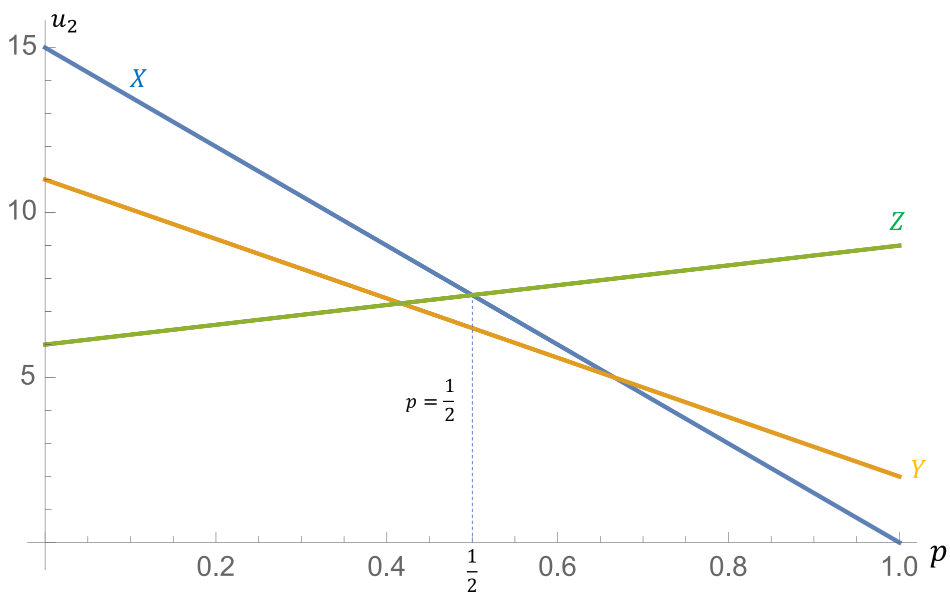
Let us look for a NE of the form

$$\begin{aligned} \sigma_1 &= pA + (1-p)B \\ \sigma_2 &= xX + yY + (1-x-y)Z \end{aligned}$$

We compute the expected utilities of the players

$$\begin{aligned} u_1(A, \sigma_2) &= 4 - 2x - 4y \\ u_1(B, \sigma_2) &= 2 + x - y \\ u_2(\sigma_1, X) &= 15 - 15p \\ u_2(\sigma_1, Y) &= 11 - 9p \\ u_2(\sigma_1, Z) &= 6 + 3p \end{aligned}$$

We graph the utilities of player 2.



The intersection of the lines $u_2(\sigma_1, X) = u_2(\sigma_1, Z)$ is given by $15 - 15 = 6 + 3p$. That is $p = \frac{1}{2}$. We see that Y is not part of any best reply for player 2. Hence, we may assume $y = 0$. And

$$\begin{aligned} \sigma_1 &= pA + (1-p)B \\ \sigma_2 &= xX + (1-x)Z \end{aligned}$$

Also, from the picture we see that best reply of player 2 is

$$\text{BR}_2(\sigma_1) = \begin{cases} X \quad (x=1) & \text{if} \quad p < \frac{1}{2} \\ \{X, Z\} \quad (0 \le x \le 1) & \text{if} \quad 0 \le p = \frac{1}{2} \\ Z \quad (x=0) & \text{if} \quad p > \frac{1}{2} \end{cases}$$

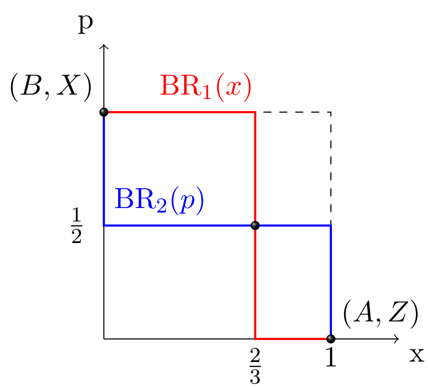
Since $y = 0$ we have that

$$\begin{aligned} u_1(A, \sigma_2) &= 4 - 2x \\ u_1(B, \sigma_2) &= 2 + x \end{aligned}$$

we have that $u_1(A, \sigma_2) \ge u_1(B, \sigma_2)$ iff $x \le \frac{2}{3}$. Thus, best reply of player 1 is

$$\text{BR}_1(\sigma_2) = \begin{cases} A \quad (p=1) & \text{if} \quad 0 \le x < \frac{2}{3} \\ \{A, B\} \quad (p \in [0,1]) & \text{if} \quad x = \frac{2}{3} \\ B \quad (p=0) & \text{if} \quad 1 \ge x > \frac{2}{3} \end{cases}$$

Graphically,



We obtain the following NE.

- $(x = 0, p = 1)$ or (A, Z), with payoffs $u_1 = 2, u_2 = 10$.
- $(x = 1, p = 0)$ or (B, X), with payoffs $u_1 = 2, u_2 = 15$.
- $\left(\frac{1}{2}A + \frac{1}{2}B, \frac{2}{3}X + \frac{1}{3}Z\right)$ with payoffs $u_1 = \frac{8}{3}$, $u_2 = \frac{15}{2}$.

2. Consider a market with one good and two firms. Firm 1 decides the price p_1 and firm 2 decides the price p_2. Given those prices, the amount sold by each company is

$$\begin{aligned} x_1(p_1, p_2) &= 84 - p_1 + \frac{p_2}{2} \\ x_2(p_1, p_2) &= 84 - p_2 + \frac{p_1}{2} \end{aligned}$$

Both firms have constant marginal cost $c = 0$.

(a) (10 points) Write the profit functions $\pi_1(p_1, p_2)$ and $\pi_2(p_1, p_2)$ of the firms.

Solution:
We assume prices are such that the quantities demanded are positive. The profits of the firms are

$$\begin{aligned}\pi_1(p_1,p_2) &= \left(84 - p_1 + \frac{p_2}{2}\right)p_1 = 84p_1 - p_1^2 + \frac{p_1p_2}{2} \\ \pi_2(p_1,p_2) &= \left(84 - p_2 + \frac{p_1}{2}\right)p_2 = 84p_2 + \frac{p_1p_2}{2} - p_2^2\end{aligned}$$

Suppose that both firms decide p_1 and p_2 simultaneous and independently.

(b) (10 points) Compute the best reply of the firms.
Solution: *The first order condition for firm 1 is*

$$84 - 2p_1 + \frac{p_2}{2}$$

The best reply function of firm 1 is

$$\mathrm{BR}_1(p_2) = \frac{p_2}{4} + 42$$

Similarly, the reply function of firm 1 is

$$\mathrm{BR}_2(p_1) = \frac{p_1}{4} + 42$$

(c) (20 points) Compute the Nash equilibrium of the **static game**. What are the profits of the firms in equilibrium?
Solution: *The Nash equilibrium (p_1^*, p_2^*) is the solution of the following system of equations*

$$\begin{aligned}p_1 &= \frac{p_2}{4} + 42 \\ p_2 &= \frac{p_1}{4} + 42\end{aligned}$$

The solution is $p_1^* = p_2^* = 56$. The equilibrium profits are $\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = 3136$.

Suppose now that firm 1 decides first p_1 and, after observing p_1, firm 2 chooses p_2.

(d) (10 points) Compute the sub-game perfect Nash equilibrium of the **dynamic game**. What is the equilibrium path? What are the profits of the firms in equilibrium?
Solution: *We use backward induction. Te best reply of firm 2 continues to be*

$$\mathrm{BR}_2(p_1) = \frac{p_1}{4} + 42$$

Anticipating this reaction, the induced utility on firm 1 is

$$u_1\left(p_1, \frac{p_1}{4} + 42\right) = 105p_1 - \frac{7p_1^2}{8}$$

The above function is maximized for $\bar{p}_1 = 60$. *Hence, the SPNE is*

$$\bar{p}_1 = 60, \quad \text{BR}_2(p_1) = \frac{p_1}{4} + 42$$

The equilibrium path is $p_1 = 60, \quad p_2 = 57$. *The profits of the firms are* $\pi_1(60, 57) = 3150$, $\pi_2(60, 57) = 3249$.

(e) (5 points) Is there a first mover or a last mover advantage in the dynamic game?
Solution: *The follower has a higher profit. There is a last mover advantage.*

(f) (10 points) Is $p_1 = 40$, $p_2 = 52$ a **Nash equilibrium of the static game**? Is there a **Nash equilibrium of the dynamic game** in which in the equilibrium path players play $p_1 = 40$, $p_2 = 52$? Is there a **sub-game perfect Nash equilibrium of the dynamic game** in which in the equilibrium path players play $p_1 = 40$, $p_2 = 52$?
You must justify each of the answers.
Solution: *We have that*

$$\text{BR}_1(52) = \frac{52}{4} + 42 = 55 \neq 40$$

Hence, $p_1 = 40$, $p_2 = 52$ *is* ***NOT*** *a NE of the static game. However, we are going to check that the following strategy profile*

$$p_1 = 40, \quad \textit{and} \quad p_2(p_1) = \begin{cases} 52 & \textit{if } p_1 = 40 \\ 0 & \textit{if } p_1 \neq 40 \end{cases}$$

is a NE of the dynamic game. The path determined by this strategy profile is $p_1 = 40$, $p_2 = 52$. *And players obtain the payoffs* $u_1(40, 52) = 2800$, $u_2(40, 52) = 2704$.
If player 1 deviates and plays $p_1 \neq 40$ *then* $p_2(p_1) = 0$ *and the payoff of player 1 is* $u(p_1, 0) = 84p_1 - p_1^2$ *which is maximized for* $p_1 = 42$. *The highest payoff player 1 can achieve with* $p_1 \neq 40$ *is* $u(42, 0) = 1764$. *Hence, player 1 does not have incentives to deviate. Since,*

$$\text{BR}_2(40) = \frac{40}{4} + 42 = 52$$

player 2 has not incentives to deviate. Therefore, $p_1 = 40$, $p_2 = 52$ *is* ***IS*** *a NE of the dynamic game. It is* ***not sub-game perfect*** *because, by backward induction, the only SPNE is the one found in part (d).*

3. Consider the following normal form game.

		Player 2	
		A	B
Player 1	A	10, 10	0, 20
	B	20, 0	1, 1

The stage game G

(a) (5 points) Find the all the Nash equilibria of the game G.

Solution: *There is a unique NE (B, B) with payofss $u_1 = u_2 = 1$. It is in dominant strategies.*

Assume now that the above stage game is played infinitely many times. After each round, players observe the moves done by the other player. The total payoffs of the repeated game are the discounted (with discount factor δ) sums of the payoffs obtained in each round.

(b) (10 points) Describe the trigger strategy in which, in the equilibrium path, (A, A) is played in every round.

Solution: *Let us consider trigger strategies: Player $i = 1, 2$ at*

- $t = 1$ *plays A;*
- $t > 1$ *plays A if (A, A) was played at $t = 1, \ldots, t-1$. Otherwise, play B.*

(c) (20 points) For what values of the discount factor δ the trigger strategy described in the previous part constitutes a Nash equilibrium of the game repeated infinitely many times?

Solution: *The payoffs obtained by both players with the trigger strategy are,*

$$u^c = 10 + 10\delta + \cdots + 10\delta^t + 10\delta^{t+1} + 10\delta^{t+2} + 10\delta^{t+3} + \cdots$$

If one player deviates at stage t and the other player follows the trigger strategy, the payoff of the player which deviates is

$$u^d = 10 + 10\delta + \cdots + 10\delta^t + 20\delta^{t+1} + \delta^{t+2} + \delta^{t+3} + \cdots$$

The trigger strategy is NE iff $u^c \geq u^d$. That is,

$$10\delta^{t+1} + 10\delta^{t+2} + 10\delta^{t+3} + \cdots \geq 20\delta^{t+1} + \delta^{t+2} + \delta^{t+3} + \cdots$$

or

$$10 + 10\delta + 10\delta^2 + \cdots \geq 20 + \delta + \delta^2 + \cdots$$

That is

$$9\delta + 9\delta^2 + \cdots \geq 10$$

which is the same as

$$\frac{9\delta}{1-\delta} \geq 10$$

Thus, the trigger strategy is a NE of the repeated game iff

$$\delta \geq \frac{10}{19}$$

(d) (10 points) For the values of the discount factor δ found in the previous part argue that the trigger strategy constitutes a subgame perfect Nash equilibrium of the game repeated infinitely many times?

Solution:

There are two types of subgames starting at a stage t.

- *Subgames in which at every stage* $1, 2, \ldots, t-1$ *it was played* (A, A)*. Then, the situation in the subgame that starts at this node is exactly as above, except that the payoffs are multiplied by* δ^{t-1}*. The above argument shows that the trigger strategy is also a NE of that subgame.*
- *Subgames in which at some stage* $1, 2, \ldots, t-1$ *the strategy profile* (A, A) *was not played. In these subgames the trigger strategy prescribes that* (C, C)*, a NE of the stage game* G*, is played. But, this is a SPNE of this subgame.*

Therefore, if $\delta \geq \frac{2}{3}$*, the trigger strategy is a SPNE of the repeated game. Note that in this strategy profile, the players cooperate at every stage of the repeated game.*

Consider now the following variant of the trigger strategy: Players cooperate until one player defects. Whenever a defection occurs, players play the NE of the stage game for k periods. After punishing for k periods, they cooperate again until a new defection occurs. Formally,

- At $t = 1$ players play (A, A).
- At any $t > 1$ if either:
  - (A, A) was played in period $t-1$; or
  - (B, B) was played in the periods $t-k, t-k+1, \ldots, t-1$;

  Then, play (A, A)
- In all other cases play (B, B).

(e) (10 points) Suppose $k = 1$ is there a value for the discount factor δ for which the above modified trigger strategy is a Nash equilibrium of the game repeated infinitely many times?

Solution:

For simplicity, we consider only deviations at $t = 1$*. The payoffs obtained by both players with the trigger strategy are,*

$$u^c = 10 + 10\delta + 10\delta^2 + \cdots$$

If one player deviates only at stage $t = 1$ *and the other player follows the trigger strategy, the payoff of the player which deviates is*

$$u^d = 20 + \delta + 10\delta^2 + 10\delta^3 + \cdots$$

The trigger strategy is NE iff $u^c \geq u^d$*. That is,*

$$10 + 10\delta \geq 20 + \delta$$

which is the same as

$$\delta \geq \frac{10}{9} > 1$$

Hence, there is no value of $0 < \delta < 1$ *for which the above modified trigger strategy is a Nash equilibrium of the game repeated infinitely many times.*

(f) (10 points) Suppose $k = 2$ is there a value for the discount factor δ for which the above modified trigger strategy is a Nash equilibrium of the game repeated infinitely many times? **Hint:** $\sqrt{441} = 21$.

Solution:

For simplicity, we consider only deviations at $t = 1$. The payoffs obtained by both players with the trigger strategy are,

$$u^c = 10 + 10\delta + 10\delta^2 + 10\delta^3 + \cdots$$

If one player deviates only at stage $t = 1$ and the other player follows the trigger strategy, the payoff of the player which deviates is

$$u^d = 20 + \delta + \delta^2 + 10\delta^3 + 10\delta^4 + \cdots$$

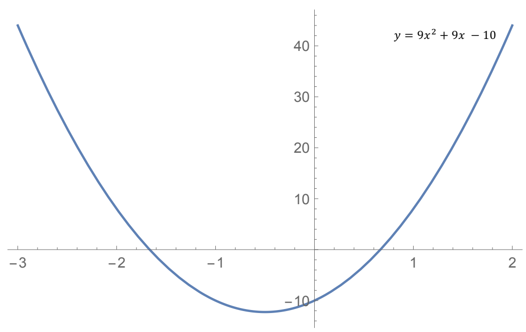
The trigger strategy is NE iff $u^c \geq u^d$. That is,

$$10 + 10\delta + 10\delta^2 \geq 20 + \delta + \delta^2$$

which is the same as

$$9\delta^2 + 9\delta - 10 \geq 0$$

Consider the parabola $y = 9x^2 + 9x - 10$. The graph of the parabola is the following.



The intersection points with the x axis are

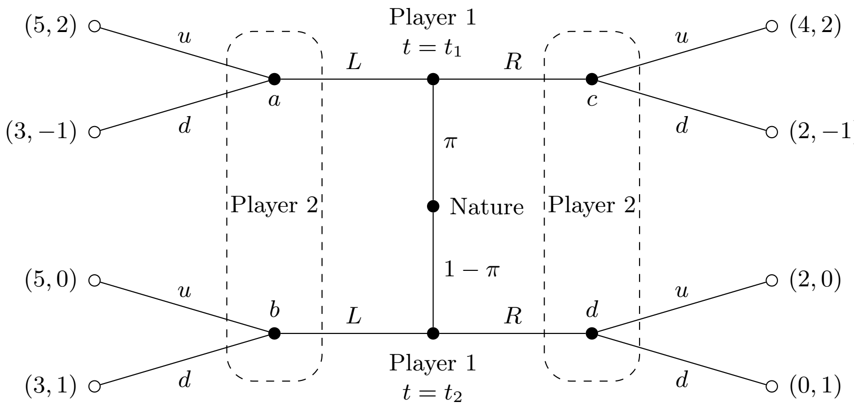
$$x_0 = \frac{-9 - \sqrt{441}}{18} = -\frac{5}{3}, \quad x_1 = \frac{-9 + \sqrt{441}}{18} = \frac{2}{3}$$

Hence, for

$$\frac{2}{3} \leq \delta < 1$$

the above modified trigger strategy is a Nash equilibrium of the game repeated infinitely many times.

4. Consider the following signalling game. There are two types of player 1, $t = t_1$ and $t = t_2$.



Hint: Justify that one of the types of player 1 has a dominant strategy. That simplifies a lot the analysis.

(a) (10 points) Compute all the separating perfect Bayesian Nash equilibria (in pure strategies) of the above game. Write the separating PBNE, including the beliefs of player 2. Justify your answer.

Solution:

- *The strategies of player 1 are* $S_1 = \{LL, LR, RL, RR\}$, *were the first letter denotes the strategy of* t_1 *and the second letter the strategy, of* t_2.
- *The strategies of the player 2 are* $S_1 = \{uu, ud, ud, uu\}$, *were the first letter denotes the strategy of* P_2 *after observing* L *and the second letter the strategy, of* P_2 *after observing* R.

The normal form game is (The Sender chooses rows and the Receiver chooses columns)

	uu	ud	du	dd
LL	$(5, 2\pi)$	$(5, 2\pi)$	$(3, 1-2\pi)$	$(3, 1-2\pi)$
LR	$(3\pi+2, 2\pi)$	$(5\pi, \pi+1)$	$(\pi+2, -\pi)$	$(3\pi, 1-2\pi)$
RL	$(5-\pi, 2\pi)$	$(5-3\pi, -\pi)$	$(\pi+3, \pi+1)$	$(3-\pi, 1-2\pi)$
RR	$(2\pi+2, 2\pi)$	$(2\pi, 1-2\pi)$	$(2\pi+2, 2\pi)$	$(2\pi, 1-2\pi)$

The best replies of the players are

a. *For* $\pi > \frac{1}{4}$

	uu	ud	du	dd
LL	$(5, 2\pi)$	$(5, 2\pi)$	$(3, 1-2\pi)$	$(3, 1-2\pi)$
LR	$(3\pi+2, 2\pi)$	$(5\pi, \pi+1)$	$(\pi+2, -\pi)$	$(3\pi, 1-2\pi)$
RL	$(5-\pi, 2\pi)$	$(5-3\pi, -\pi)$	$(\pi+3, \pi+1)$	$(3-\pi, 1-2\pi)$
RR	$(2\pi+2, 2\pi)$	$(2\pi, 1-2\pi)$	$(2\pi+2, 2\pi)$	$(2\pi, 1-2\pi)$

$\pi > \frac{1}{4}$

We obtain the NE in pure strategies: (LL, uu), (LL, ud) *and* (RL, du).

b. *For* $\pi < \frac{1}{4}$

	uu	ud	du	dd
LL	$(5, 2\pi)$	$(5, 2\pi)$	$(3, 1-2\pi)$	$(3, 1-2\pi)$
LR	$(3\pi+2, 2\pi)$	$(5\pi, \pi+1)$	$(\pi+2, -\pi)$	$(3\pi, 1-2\pi)$
RL	$(5-\pi, 2\pi)$	$(5-3\pi, -\pi)$	$(\pi+3, \pi+1)$	$(3-\pi, 1-2\pi)$
RR	$(2\pi+2, 2\pi)$	$(2\pi, 1-2\pi)$	$(2\pi+2, 2\pi)$	$(2\pi, 1-2\pi)$

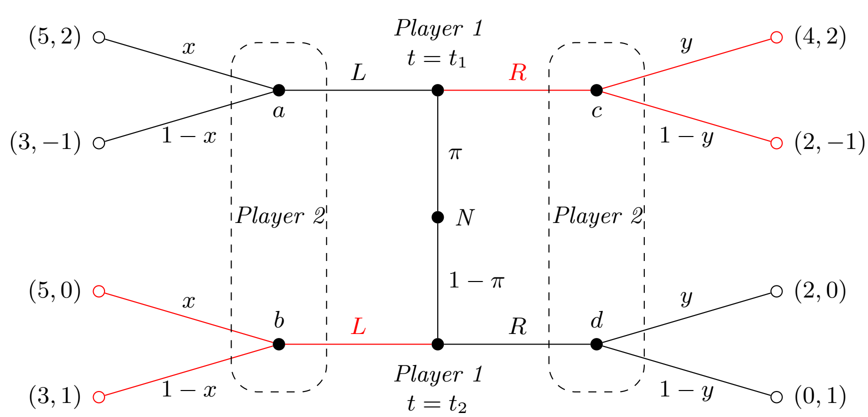
$\pi < \frac{1}{4}$

We obtain the NE in pure strategies: (RL, du) *and* (LL, dd).

Which of the above NE are PBNE? Note that the minimum payoff of player t_2 *with* L *is* 3. *Whereas the maximum payoff of player 2 with* R *is* 2. *Hence, in any PBNE* $m(t_2) = L$.

A separating PBNE

The only candidate for a separating PBNE is of the form $(RL, **)$

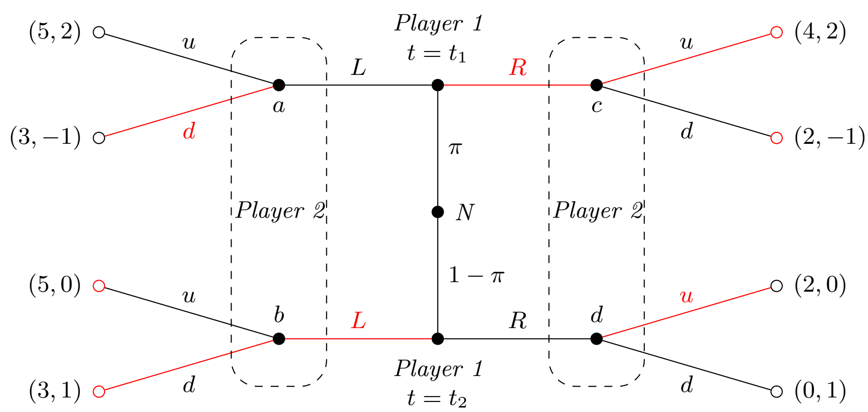


The Beliefs of player 2 are

$$\mu_2(a|L) = 0, \quad \mu_2(b|L) = 1$$
$$\mu_2(c|R) = 1, \quad \mu_2(d|R) = 0$$

Given the above beliefs, the best replies of player 2 are

$$\text{BR}_2(L) = d, \quad \text{BR}_2(R) = u$$



The utilities of player 1 are

$$u_1 (L|t_1) = 3, \quad u_1 (R|t_1) = 4$$

$$u_1 (L|t_2) = 3, \quad u_1 (R|t_2) = 2$$

If t_1 switches from R to L the payoff goes down from 4 to 3. If t_2 switches from L to R the payoff goes down from 3 to 2. Thus, Player 1 does not want to switch his strategy. Therefore, (RL, du) is a ***separating*** *PBE with the Beliefs*

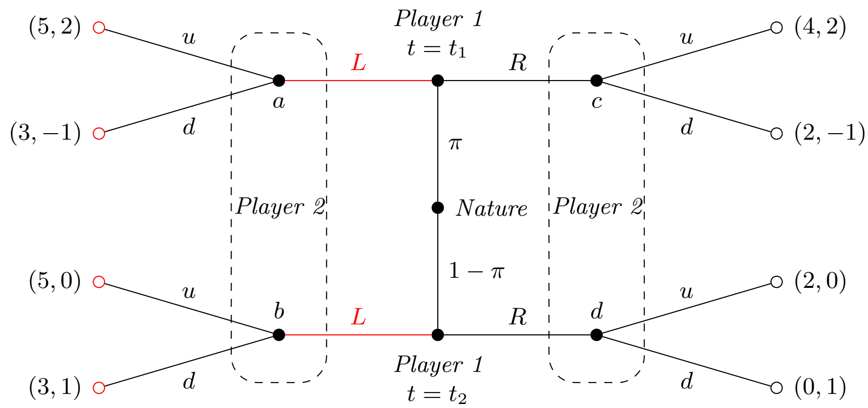
$$\begin{aligned} \mu_2(a|L) &= 0, \quad \mu_2(b|L) = 1 \\ \mu_2(c|R) &= 1, \quad \mu_2(d|R) = 0 \end{aligned}$$

The payoffs of the players are

$$\begin{aligned} u_1 (t_1) &= 4, \quad u_1 (t_2) = 3 \\ u_1 &= 4 \times \pi + 3 \times (1 - \pi) = 3 + \pi \\ u_2 &= 2 \times \pi + 1 \times (1 - \pi) = 1 + \pi. \end{aligned}$$

(b) (10 points) Compute all the pooling perfect Bayesian Nash equilibria (in pure strategies) of the above game. Write the pooling PBNE, including the beliefs of player 2. Justify your answer.

Solution: *The only candidate for a pooling PBNE is of the form $(LL, **)$.*



The Beliefs of player 2 are

$$\begin{aligned} \mu_2(a|L) &= \pi, \quad \mu_2(b|L) = 1-\pi \\ \mu_2(c|R) &= z, \quad \mu_2(d|R) = 1-z \end{aligned}$$

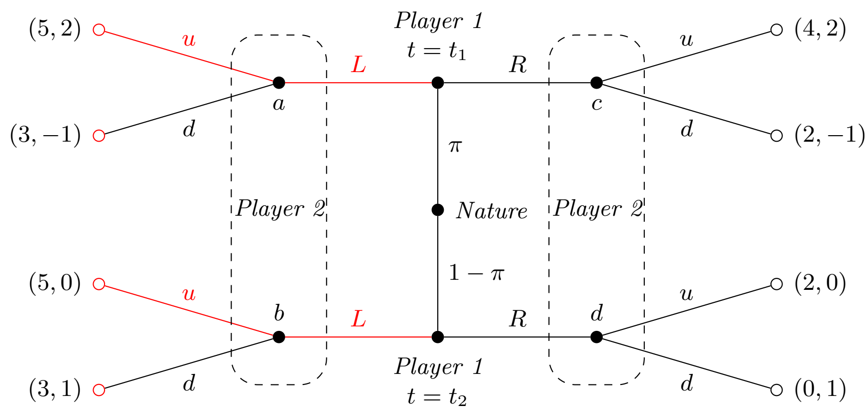
Given the above beliefs, the expected utility of player 2 after observing L is

$$\begin{aligned} u_2(u|L) &= 2\pi + 0 \times (1-\pi) = 2\pi \\ u_2(d|L) &= -\pi + 1 \times (1-\pi) = 1-2\pi \end{aligned}$$

Hence,

$$\mathrm{BR}_2(L) = \begin{cases} u & \textit{if } \pi > \frac{1}{4} \\ \{u, d\} & \textit{if } \pi = \frac{1}{4} \\ d & \textit{if } \pi < \frac{1}{4} \end{cases}$$

Let us assume $\pi \geq \frac{1}{4}$, *then* $\mathrm{BR}_2(L) = u$. *Graphically,*



Now $u_1(L|t_i) = 5$ *for* $i = 1, 2$, *the maximum possible for player 1. Hence,* $m(t_i) = L$ *for* $i = 1, 2$

On the other hand, the expected utility of player 2 after observing R is

$$\begin{aligned} u_2(u|R) &= 2\times z + 0\times(1-z) = 2z \\ u_2(d|R) &= -1\times z + 1\times(1-z) = 1-2z \end{aligned}$$

Hence, we obtain the following PBNE, for $\pi \geq \frac{1}{4}$.

a. (LL, uu) *with with the Beliefs*

$$\begin{aligned} \mu_2(a|L) &= \pi, \quad \mu_2(b|L) = 1-\pi \\ \mu_2(c|R) &= z, \quad \mu_2(d|R) = 1-z, \quad z \geq \frac{1}{4} \end{aligned}$$

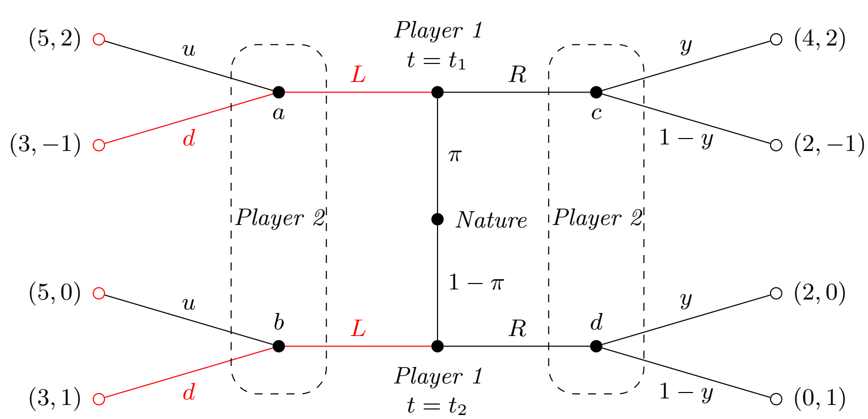
b. (LL, ud) *with with the Beliefs*

$$\begin{aligned} \mu_2(a|L) &= \pi, \quad \mu_2(b|L) = 1-\pi \\ \mu_2(c|R) &= z, \quad \mu_2(d|R) = 1-z, \quad z \leq \frac{1}{4} \end{aligned}$$

In both of the above BNE the payoffs of the players are

$$\begin{aligned} u_1(t_1) &= u_1(t_2) = 5 \\ u_1 &= 5 \\ u_2 &= 2\times\pi + 0\times(1-\pi) = 2\pi. \end{aligned}$$

Let us assume $\pi \leq \frac{1}{4}$, *then* $\mathrm{BR}_2(L) = d$. *Graphically,*



Now $u_1(L|t_2) = 3$. *If player* t_2 *deviates to* R *the maximum payoff is* 2. *Therefore,* $m(t_2) = L$. *On the other hand, we have* $u_1(L|t_1) = 3$. *And* $m(t_1) = L$ *if and only if* $\mathrm{BR}_2(R) = d$: *Now, the expected utility of player 2 after observing R is*

$$\begin{aligned} u_2(u|R) &= 2\times z + 0\times(1-z) = 2z \\ u_2(d|R) &= -1\times z + 1\times(1-z) = 1-2z \end{aligned}$$

And $\mathrm{BR}_2(R) = d$ *if and only if* $z \leq \frac{1}{4}$. *Hence, for* $\pi \leq \frac{1}{4}$, *we obtain the PBNE* (LL, dd), *with the Beliefs*

$$\begin{aligned}\mu_2(a|L) &= \pi, \quad \mu_2(b|L) = 1-\pi \\ \mu_2(c|R) &= z, \quad \mu_2(d|R) = 1-z, \quad z \leq \frac{1}{4}\end{aligned}$$

The payoffs of the players are

$$\begin{aligned}u_1(t_1) &= u_1(t_2) = 3 \\ u_1 &= 3 \\ u_2 &= (-1)\times\pi + 1\times(1-\pi) = 1-2\pi.\end{aligned}$$

(c) (10 points) Compute all the Bayesian Nash equilibria in mixed strategies of the above game, including the beliefs of player 2. Justify your answer.

Solution: *We look for a PBE in mixed strategies of the form*

$$\begin{aligned}m(t_1) &= qL + (1-q)R, \quad 0 \leq q \leq 1 \\ m(t_2) &= L \\ a_2(L) &= xu + (1-x)d \quad 0 \leq x \leq 1 \\ a_2(R) &= yu + (1-y)d \quad 0 \leq y \leq 1\end{aligned}$$

Is there a PBE with $0 < y < 1$?

*From the computation of the PBE (LL, dd) we have seen that for $\pi \leq \frac{1}{4}$ and $\mu_2(c|R) = \frac{1}{4}$, player 2, after observing R is indifferent between u and d. And we have also seen that $m(t_1) = L$ if and only if $y \leq \frac{1}{2}$. Therefore, we obtain the following **pooling PBE**.*

$$\begin{aligned}m(t_1) &= L \\ m(t_2) &= L \\ a_2(L) &= d \\ a_2(R) &= yu + (1-y)d \quad 0 \leq y \leq \frac{1}{2}\end{aligned}$$

with the beliefs

$$\begin{aligned}\mu_2(a|L) &= \pi, \quad \mu_2(b|L) = 1-\pi, \quad \pi \leq \frac{1}{4} \\ \mu_2(c|R) &= \frac{1}{4}, \quad \mu_2(d|R) = \frac{3}{4}\end{aligned}$$

and the payoffs

$$\begin{aligned} u_1(t_1) &= u_1(t_2) = 3 \\ u_1 &= 3 \\ u_2 &= (-1)\times\pi + 1\times(1-\pi) = 1-2\pi. \end{aligned}$$

Is a PBNE with $0 < q < 1$, $0 \le x \le 1$?

In such a case

$$\begin{aligned} u_1(L|t_1) &= 5x + 3(1-x) = 3+2x \\ u_1(R|t_1) &= 4y + 2(1-y) = 2+2y \end{aligned}$$

So, we have $3+2x = 2+2y$. *That is,* $y - x = \frac{1}{2}$. *It follows that*

$$0 \le x \le \frac{1}{2} \le y \le 1$$

The beliefs of player 2 are,

$$\begin{aligned} \mu_2(a|L) &= \frac{\pi q}{\pi q + 1 - \pi}, \quad \mu_2(b|L) = \frac{1-\pi}{\pi q + 1 - \pi} = \\ \mu_2(c|R) &= 1, \quad \mu_2(d|R) = 0 \end{aligned}$$

Hence, $m_2(R) = u$, *that is* $y = 1$ *and* $x = \frac{1}{2}$. *Now we have,*

$$\begin{aligned} u_2(u|L) &= 2\frac{\pi q}{\pi q + 1 - \pi} + 0 \times \frac{1-\pi}{\pi q + 1 - \pi} = \frac{2\pi q}{\pi q + 1 - \pi} \\ u_2(d|L) &= (-1)\times\frac{\pi q}{\pi q + 1 - \pi} + 1 \times \frac{1-\pi}{\pi q + 1 - \pi} = \times\frac{1-\pi-\pi q}{\pi q + 1 - \pi} \end{aligned}$$

Therefore, $2\pi q = 1 - \pi - \pi q$, *so*

$$q = \frac{1-\pi}{3\pi}$$

and $q < 1$ *if and only if* $\pi > \frac{1}{4}$. *We obtain the following PBE,*

$$\begin{aligned} m(t_1) &= qL + (1-q)R, \quad q = \frac{1-\pi}{3\pi} \\ m(t_2) &= L \\ a_2(L) &= \frac{1}{2}u + \frac{1}{2}d \\ a_2(R) &= u \end{aligned}$$

with the beliefs

$$\begin{aligned}\mu_2(a|L) &= \frac{\pi q}{\pi q + 1 - \pi}, \quad \mu_2(b|L) = \frac{1-\pi}{\pi q + 1 - \pi} = \\ \mu_2(c|R) &= 1, \quad \mu_2(d|R) = 0\end{aligned}$$

and the payoffs

$$\begin{aligned}u_1(t_1) &= u_1(t_2) = 4 \\ u_1 &= 4 \\ u_2 &= \frac{2\pi q}{\pi q + 1 - \pi} = \frac{1}{2}\end{aligned}$$