Name (Print): ____________________

University Carlos III de Madrid

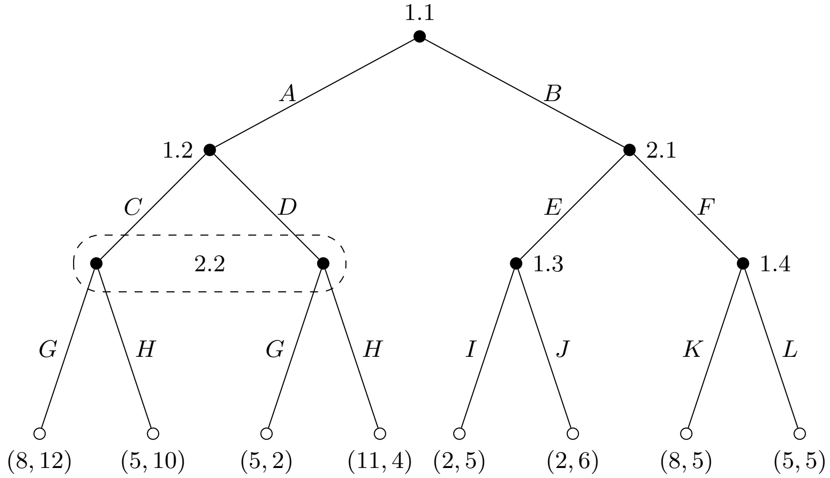
Master in Economics
Master in Industrial Economics and Markets

Final Exam. Game Theory. **01/10/2024**.

Time Limit: 120 Minutes.

Exercise	Points	Score
1	35	
2	20	
3	35	
4	70	
Total:	160	

1. Consider the following game in extensive form



(a) (5 points) What are the sub-games of the above game? It is enough to write the node at which each sub-game starts.
Solution: *There are five sub-games that start at the nodes 1.1, 1.2 2.1, 1.3 and 1.4.*

(b) (10 points) Write the normal form of the sub-game that starts at at node 1.2. Find the Nash equilibria (in pure and mixed strategies) of this sub-game.
Solution: *The normal form of the sub-game that starts at at node* 1.2 *is,*

	G	H
C	8, 12	5, 10
D	5, 2	11, 4

There are two NE in pure strategies: (C, G) *with payoffs* $(8, 12)$ *and* (D, H) *with payoffs* $(11, 4)$. *In addition, there is mixed strategy NE*

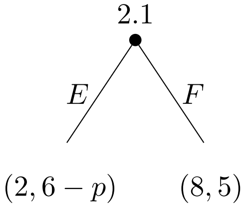
$$\left(\left(\frac{1}{2}C + \frac{1}{2}D\right), \left(\frac{2}{3}G + \frac{1}{3}H\right)\right)$$

with payoffs

$$u_1 = u_2 = 7$$

(c) (20 points) Write the subgame perfect Nash equilibria of the whole game

Solution: *At node* 1.4 *player 1 chooses* K *with payoffs* $(u_1, u_2) = (5, 5)$. *At node* 1.3 *player 1 is indifferent between the strategies* I *and* J. *So, at this node the SPNE are of the form* $pI + (1-p)J$ *with payoffs* $(u_1, u_2) = (2, 6-p)$ *for* $0 \leq p \leq 1$.



So, at node 2.1,

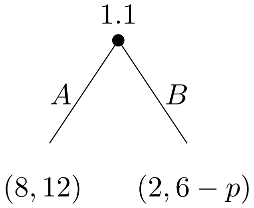
- *For* $0 \leq p < 1$, *player 2 chooses* E, *with payoffs* $(u_1, u_2) = (2, 6-p)$.
- *For* $p = 1$, *player 2 is indifferent between* E *and* F *and chooses* $qE + (1-q)F$ *with* $0 \leq q \leq 1$, *with payoffs* $(u_1, u_2) = (8 - 6q, 5)$.

We use the notation $(1.1, 1.2, 2.1, 1.3, 1.4)$. *All the SPNE are of the form,*

- $(*, *, I, E, pI + (1-p)J, K)$, *for* $0 \leq p < 1$, *with payoffs* $(u_1, u_2) = (2, 6-p)$.
- $(*, *, qE + (1-q)F, I, K)$, $0 \leq q \leq 1$, *with payoffs* $(u_1, u_2) = (8 - 6q, 5)$.

a. *Let us look for SPN in which in the subgame that starts at* 1.2 *the NE* (C, G) *is played. There are two families of SPNE.*

   (a) *For* $0 \leq p < 1$:

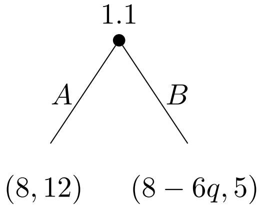


   We obtain the SPNE

$$(A, (C, G), I, E, pI + (1-p)J, K), \quad 0 \leq p < 1$$

   with payoffs $(u_1, u_2) = (8, 12)$.

   (b) *For* $p = 1$:



i. *For $q = 0$, player 1 is indifferent between A and B. We obtain the SPNE*

$$(xA + (1-x)B, (C, G), F, I, K), \quad 0 \le x \le 1$$

with payoffs $(u_1, u_2) = (8, 5 + 7x)$.

ii. *For $0 < q \le 1$, we obtain the SPNE*

$$(A, (C, G), qE + (1-q)F, I, K), \quad 0 < q \le 1$$

with payoffs $(u_1, u_2) = (8, 12)$.

b. *Let us look for SPN in which in the subgame that starts at 1.2 the NE (D, H) is played. We obtain the SPNE*

$$(A, (D, H), I, E, pI + (1-p)J, K), \quad 0 \le p < 1$$

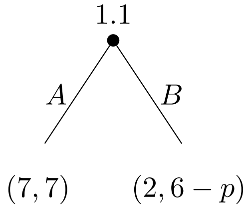
with payoffs $(u_1, u_2) = (11, 4)$ and

$$(A, (D, H), qE + (1-q)F, I, K), \quad 0 \le q \le 1$$

with payoffs $(u_1, u_2) = (11, 4)$.

c. *Let us look for SPN in which in the subgame that starts at 1.2 the NE $\left(\left(\frac{1}{2}C + \frac{1}{2}D\right), \left(\frac{2}{3}G + \frac{1}{3}H\right)\right)$ is played.*

(a) *For $0 \le p < 1$:*

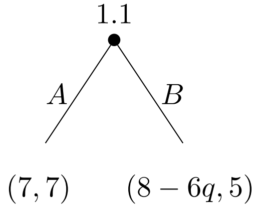


We obtain the SPNE

$$\left(A, \left(\left(\frac{1}{2}C + \frac{1}{2}D\right), \left(\frac{2}{3}G + \frac{1}{3}H\right)\right), E, pI + (1-p)J, K\right), \quad 0 \le p < 1$$

with payoffs $(u_1, u_2) = (7, 7)$.

(b) *For $p = 1$:*



i. *For $q = \frac{1}{6}$, player 1 is indifferent between A and B. We obtain the SPNE*

$$\left(xA + (1-x)B, \left(\left(\frac{1}{2}C + \frac{1}{2}D\right), \left(\frac{2}{3}G + \frac{1}{3}H\right)\right), \frac{1}{6}E + \frac{5}{6}F, I, K\right), \quad 0 \le x \le 1$$

with payoffs $(u_1, u_2) = (7, 5 + 2x)$.

ii. *For $0 < q \leq \frac{1}{6}$, we obtain the SPNE*

$$\left(B, \left(\left(\frac{1}{2}C + \frac{1}{2}D\right), \left(\frac{2}{3}G + \frac{1}{3}H\right)\right), qE + (1-q)F, I, K\right), \quad 0 < q \leq \frac{1}{6}$$

with payoffs $(u_1, u_2) = (8 - 6q, 5)$.

iii. *For $\frac{1}{6} < q \leq 1$, we obtain the SPNE*

$$\left(A, \left(\left(\frac{1}{2}C + \frac{1}{2}D\right), \left(\frac{2}{3}G + \frac{1}{3}H\right)\right), qE + (1-q)F, I, K\right), \quad \frac{1}{6} < q \leq 1$$

with payoffs $(u_1, u_2) = (7, 7)$.

2. Two risk averse individuals with utility functions $u(x) = x^{1/3}$, where x represents money, face a first price auction. Agent $i = 1, 2$ ($i = 1, 2$) values the good in v_i monetary units. This valuation is private information, but it is known that the v_i's are random variables independently and uniformly distributed in the interval $[0, 1]$.

   (a) (10 points) Describe the situation as a Bayesian game.

   Solution: *There are two players, $N = \{1, 2\}$. Their types are $T_1 = T_2 = [0, 1]$. The sets of strategies are $S_i(v_i) \in [0, v_i], i = 1, 2$. The beliefs of the players are*

$$p_1(v_2 \leq c | v_1) = p_2(v_1 \leq c | v_2) = \begin{cases} 0 & \text{if } c < 0 \\ c & \text{if } c \in [0, 1] \\ 1 & \text{if } c > 1 \end{cases},$$

   The utilities of the players are

$$u_i(b_1, b_2; v_i) = \begin{cases} 0 & \text{if } b_i < b_j \\ \frac{(v_i - b_i)^{\frac{1}{3}}}{2} & \text{if } b_i = b_j \\ (v_i - b_i)^{\frac{1}{3}} & \text{if } b_i > b_j \end{cases} \quad i = 1, 2, \quad i \neq j$$

   (b) (10 points) Find a bayesian Nash equilibrium of the form $b_i(v_i) = \alpha_i v_i$. What is the utility of each individual in equilibrium?

   Solution: *Suppose player 2 follows the strategy $b_2(v_2) = \alpha_2 v_2$. If player 1 chooses to bid b_1, his expected utility is*

$$\begin{aligned} u_1(b_1|v_1) &= p(b_1 > b_2(v_2))(v_1 - b_1)^{\frac{1}{3}} + p(b_1 = b_2(v_2))\frac{1}{2}(v_1 - b_1)^{\frac{1}{3}} + 0 \times p(b_1(v_1) < b_2(v_2)) \\ &= (v_1 - b_1)^{\frac{1}{3}} p(b_1 > b_2(v_2)) \end{aligned}$$

   because $p(b_1 = b_2(v_2)) = 0$. Thus,

$$\begin{aligned} u_1(b_1|v_1) &= p(b_1 > b_2(v_2))(v_1 - b_1)^{\frac{1}{3}} = p(b_1 > \alpha_2 v_2)(v_1 - b_1)^{\frac{1}{3}} \\ &= p\left(v_2 < \frac{b_1}{\alpha_2}\right)(v_1 - b_1)^{\frac{1}{3}} = \frac{b_1 (v_1 - b_1)^{\frac{1}{3}}}{\alpha_2} \end{aligned}$$

The best reply of player 1 is given my the solution to

$$\max_{b_1} b_1 \left(v_1 - b_1\right)^{\frac{1}{3}}$$

The first order condition is

$$\left(v_1 - b_1\right)^{\frac{1}{3}} = \frac{b_1}{3\left(v_1 - b_1\right)^{\frac{2}{3}}}$$

whose solution is

$$b_1(v_1) = \frac{3}{4} v_1$$

Similarly, if player 1 follows the strategy $b_1(v_1) = \alpha_1 v_1$ *the best reply for player 2 is*

$$b_2(v_2) = \frac{3}{4} v_2$$

Hence

$$b_i(v_i) = \frac{3}{4} v_i, \quad i = 1, 2$$

is a BNE. The expected payoff of agent $i = 1, 2$ *is*

$$u_i\left(b_1(v_1), b_2(v_2); v_i\right) = \frac{\frac{3}{4} v_i \left(v_i - \frac{3}{4} v_i\right)^{\frac{1}{3}}}{\frac{3}{4}} = \frac{1}{4^{\frac{1}{3}}} v_i^{\frac{4}{3}}$$

3. Consider the situation in which player 2 knows which game is played (a or b below). However, player 1 only knows that table a is played with probability $\frac{1}{3}$ and table b is played with probability $\frac{1}{2}$.

		Player 2	
		C	*D*
Player 1	*A*	18 ,42	6, 6
	B	12, 6	24,18

a

		Player 2	
		C	*D*
Player 1	*A*	30 ,6	6, 18
	B	24, 30	12, 6

b

(a) (5 points) Describe the situation as a Bayesian game.

Solution: *There are two players* $N = \{1, 2\}$*. There are two types of player 2:* $T_2 = \{a, b\}$*. There is one type of player 1:* $T_1 = \{t\}$*. The sets of strategies are* $S_2 = \{CC, CD, DC, DD\}$, $S_1 = \{A, B\}$*. The beliefs of the players are*

$$\begin{aligned} p_2\left(t_1 = t | t_2 = a\right) &= p_2\left(t_1 = t | t_2 = b\right) = 1 \\ p_1\left(t_2 = a | t_1 = t\right) &= 1/3, \quad p_1\big(t_2 = c_| t_1 = t\big) = 2/3 \end{aligned}$$

The payoffs are given by the above tables.

(b) (10 points) Find the Bayesian–Nash equilibria in pure strategies and the payoffs of the players.

Solution: *The associated normal form game is*

	CC	CD	DC	DD
A	$(26,18)$	$(10,26)$	$(22,6)$	$(6,14)$
B	$(20,22)$	$(12,6)$	$(24,26)$	$(16,10)$

and we see that there is a BNE in pure strategies (B, DC) *with payoffs* $(24,26)$.

(c) (20 points) Find the Bayesian–Nash equilibria in mixed strategies and the payoffs of the players.

Solution: *Let us look for a BNE of the form*

$$(xA+(1-x)B,(yC+(1-y)D,zC+(1-z)D))$$

Let

$$\begin{aligned} s_1 &= xA+(1-x)B \\ s_a &= yC+(1-y)D \\ s_b &= zC+(1-z)D \end{aligned}$$

We have that

$$\begin{aligned} u_1(A;s_a,s_b) &= \frac{1}{3}(18y+6-6y)+\frac{2}{3}(30z+6-6z) = 6+4y+16z \\ u_1(B;s_a,s_b) &= \frac{1}{3}(12y+24(1-y))+\frac{2}{3}(24z+12(1-z)) = 16+8z-4y \\ u_a(s_1,C) &= 6+36x \\ u_a(s_1,D) &= 18-12x \\ u_b(s_1,C) &= 30-24x \\ u_b(s_1,D) &= 6+12x \end{aligned}$$

Suppose first that player 2a is using a completely mixed strategy. Then $u_a(s_1,C) = u_a(s_1,D)$. *Hence,* $6+36x = 18-12x$ *and we conclude that* $x = \frac{1}{4}$. *For this value of* x *we have that* $u_b(s_1,C) = (30-24x)|_{x=\frac{1}{4}} = 24$ *and* $u_b(s_1,D) = (6+12x)|_{x=\frac{1}{4}} = 9$, *so* $z=1$. *We check if there is a BNE of the form*

$$\left(\frac{1}{4}A+\frac{3}{4}B;(yC+(1-y)D,C)\right)$$

Player 1 must be indifferent between A *and* B. *Hence,* $6+4y+16z = 16+8z-4y$. *Since* $z=1$, *we obtain that* $y=\frac{1}{4}$. **And we have checked that**

$$\left(\frac{1}{4}A+\frac{3}{4}B;\left(\frac{1}{4}C+\frac{3}{4}D,C\right)\right)$$

is BNE in mixed strategies with payoffs $u_1 = 23$, $u_a = 15$, $u_b = 24$.

Suppose now that player 2b is using a completely mixed strategy. Then $u_b(s_1,C) = u_b(s_1,D)$. *Hence,* $30-24x = 6+12x$ *and we conclude that* $x=\frac{2}{3}$. *For this value of* x *we have that* $u_a(s_1,C) = (6+36x)|_{x=\frac{2}{3}} = 30$ *and* $u_a(s_1,D) = (18-12x)|_{x=\frac{2}{3}} = 10$. *So,*

$y = 1$. *We check if there is a BNE of the form*

$$\left(\frac{2}{3}A + \frac{1}{3}B; (C, zC + (1-z)D)\right)$$

Player 1 must be indifferent between A and B. Hence, $1 + y + 2z = 3 - y + z$. *Since* $y = 1$, *we obtain that* $z = \frac{1}{4}$. **And we have checked that**

$$\left(\frac{2}{3}A + \frac{1}{3}B; \left(C, \frac{1}{4}C + \frac{3}{4}D\right)\right)$$

is the other BNE in mixed strategies with payoffs $u_1 = 14$**,** $u_a = 30$**,** $u_b = 14$**.**

4. Consider a market with one good and two firms $i = 1, 2$. The inverse demand function is $p = 60 - q_1 - q_2$ where q_i is the quantity chosen by firm $i = 1, 2$. In the first stage, firm 1 chooses the quantity q_1. After observing q_1, firm 2 chooses q_2.

   Firm 1 has constant marginal cost $c_1 = 9$. Both firms know c_1. Firm 1 does not know the cost of firm 2. Firm 1 thinks that with probability $\frac{1}{3}$ firm 2 has constant marginal cost $c_h = 12$ and with probability $\frac{2}{3}$ firm 2 has constant marginal cost $c_l = 6$. Firm 2 knows its costs and the costs of firm 1. This situation is common knowledge for both firms.

   (a) (10 points) Describe the situation as a Bayesian game.

   Solution: *There are two players* $N = \{1, 2\}$. *There are two types of player 2:* $T_2 = \{c_l, c_h\}$, *where* $c_l = 6$ *and* $c_h = 12$. *There is one type of player 1:* $T_1 = \{c\}$. *The sets of strategies are* $S_1 = [0, \infty)$, $S_2 = \{s_2 : S_1 \to [0, \infty)\}$. *The beliefs of the players are*

$$\begin{aligned} p_2\left(t_1 = c | t_2 = c_l\right) &= p_2\left(t_1 = c | t_2 = c_l\right) = 1 \\ p_1\left(t_2 = c_l | t_1 = t\right) &= 2/3, \quad p_1\left(t_2 = c_| t_1 = t\right) = 1/3 \end{aligned}$$

   The payoffs are

$$\begin{aligned} u_h\left(q_h, q_1\right) &= q_h(60 - q_1 - q_h) - 12q_h \\ u_l\left(q_l, q_1\right) &= q_l(60 - q_1 - q_l) - 6q_l \\ u_1(q_1, q_l, q_h) &= \frac{1}{3}\left(60 - q_1 - q_h\right) q_1 + \frac{2}{3}\left(60 - q_1 - q_l\right) q_1 - 9q_1 \end{aligned}$$

(b) (10 points) Compute the best reply of each type of firm 2.

Solution: *Agent 2, type c_h, maximizes* $\max_{q_h} u_h = q_h(60-q_1-q_h)-12q_h$. *The first order condition is*

$$48 - q_1 - 2q_h = 0.$$

Note that the second derivative with respect to q_h is

$$\frac{\partial^2 u_h}{\partial q_h^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_h. The best reply of agent 2, type c_h, is

$$\mathrm{BR}_h(p_1) = \left\{0, \frac{48-q_1}{2}\right\}$$

Likewise, agent 2, type c_l, maximizes $\max_{p_l} u_l = q_l(60-q_1-q_l)-6q_l$. *The first order condition is*

$$54 - q_1 - 2q_l = 0.$$

Note that the second derivative with respect to p_l is

$$\frac{\partial^2 u_l}{\partial p_l^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_l. The best reply of Agent 2, type c_l, is

$$\mathrm{BR}_l(p_1) = \left\{0, \frac{54-q_1}{2}\right\}$$

(c) (10 points) Compute the Bayes–Nash equilibrium, the quantities sold in this equilibrium and the profits of each firm.

Solution: *Agent 1 anticipates the behavior of the two types of agent 2 and so it anticipates the utility*

$$u_1(p_1) = \frac{1}{3}\left(\frac{q_1-54}{2} - q_1 + 60\right)q_1 + \frac{2}{3}\left(\frac{q_1-48}{2} - q_1 + 60\right)q_1 - 9q_1$$

The first order condition is

$$\frac{1}{3}\left(\frac{q_1-54}{2} - q_1 + 60\right) + \frac{2}{3}\left(\frac{q_1-48}{2} - q_1 + 60\right) - \frac{q_1}{2} - 9 = 0$$

Note that

$$\frac{\partial^2 u_1}{\partial q_1^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_1. The solution is $q_1 = 26$. The BNE is

$$q_1 = 25, \quad q_l = \frac{54-q_1}{2}, \quad q_h = \frac{48-q_1}{2}$$

The equilibrium path is

$$q_1 = 25, \quad , q_l = \frac{23}{2} = 11.5, \quad q_h = \frac{29}{2} = 14.5$$

And the profits of the companies are

$$\Pi_1 = \frac{625}{2} = 312.5, \quad , \Pi_h = \frac{529}{4} = 132.25, \quad \Pi_l = \frac{841}{4} = 210.25$$

(d) (10 points) If firm 1 knew that the cost of firm 2 were $c_h = 12$, what would be the Nash equilibrium, the quantities sold in equilibrium and the profits of each firm.

Solution: *Now agent 1 anticipates that only firm 2 type c_h will be in the market and so it anticipates the utility*

$$u_1(p_1) = \left(\frac{q_1 - 48}{2} - q_1 + 60\right) q_1 - 9q_1$$

The first order condition is

$$\frac{q_1 - 48}{2} - \frac{3q_1}{2} + 51 = 0$$

Note that

$$\frac{\partial^2 u_1}{\partial q_1^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_1. The solution is $q_1 = 27$. The BNE is

$$q_1 = 27, \quad , q_h = \frac{48 - q_1}{2}$$

The equilibrium path is

$$q_1 = 27, \quad , q_h = \frac{21}{2}$$

And the profits of the companies are

$$\Pi_1 = \frac{729}{2} = 364.5, \quad \Pi_h = \frac{441}{4} = 110.25$$

(e) (10 points) If firm 1 knew that the cost of firm 2 were $c_l = 6$, what would be the Nash equilibrium, the quantities sold in equilibrium and the profits of each firm.

Solution: *Now agent 1 anticipates that only firm 2 type c_l will be in the market and so it anticipates the utility*

$$u_1(p_1) = \left(\frac{q_1 - 54}{2} - q_1 + 60\right) q_1 - 9q_1$$

The first order condition is

$$\frac{q_1 - 54}{2} - \frac{3q_1}{2} + 51 = 0$$

Note that

$$\frac{\partial^2 u_1}{\partial q_1^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_1. *The solution is* $q_1 = 24$. *The BNE is*

$$q_1 = 24, \quad , q_l = \frac{54 - q_1}{2}$$

The equilibrium path is

$$q_1 = 24, \quad , q_l = 15$$

And the profits of the companies are

$$\Pi_1 = 288, \quad \Pi_l = 224$$

(f) (10 points) We go back to the original situation in which Firm 1 does not know the cost of firm 2. Which one of the types of firm 2 prefers to reveal credibly its cost to firm 1.

Solution: *Firm 2 type* c_l *prefers to reveal its cost. Firm 2 type* c_h *prefers to hide its cost.*

(g) (10 points) Suppose now that firm 1 can pay an insider in firm 2 to find out the true cost of firm 2. Would it pay anything to know this information? If so, how much?

Solution: *Firm 1 anticipates that*

- *With probability* $\frac{1}{3}$ *firm 2 has cost* $c_h = 12$ *and firm 1 will obtain a profit of* 364.5.
- *with probability* $\frac{2}{3}$ *firm 2 has cost* $c_l = 6$ *and firm 1 will obtain a profit of* 288.

Hence, the expected profit of having the information is

$$\frac{1}{3} \times 210.25 + \frac{2}{3} \times 288 = \frac{627}{2}$$

Hence, the value of the information is

$$\frac{627}{2} - \frac{625}{2} = 1$$